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ON THE COMPLEX STRUCTURE OF KÄHLER MANIFOLDS WITH NONNEGATIVE CURVATURE

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Abstract

We study the asymptotic behavior of the Kähler-Ricci flow on Kähler manifolds of nonnegative holomorphic bisectional curvature. Using these results we prove that a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature and maximal volume growth is biholomorphic to complex Euclidean space $\mathbb{C}^n$. We also show that the volume growth condition can be removed if we assume the Kähler manifold has average quadratic scalar curvature decay and positive curvature operator.

1. Introduction

The classical uniformization theorem says that a simply connected Riemann surface is either the Riemann sphere, the open unit disk or the complex plane. On the other hand, there is a close relation between the complex structure and the geometry of a Riemann surface. An important case of this is that a complete noncompact Riemannian surface with positive Gaussian curvature is necessarily conformally equivalent to the complex plane. In higher dimensions, there is a long standing conjecture predicting similar results. In its most general form, the conjecture is due to Yau [**43**], and it states: *A complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to* $\mathbb{C}^n$. In fact, the conjecture is part of a program proposed by Yau in 1974 to study complex manifolds of parabolic type, see [**43**].

The first result supporting this conjecture was due to Mok-Siu-Yau [**28**]. There, the authors proved that if M^n is a complete noncompact Kähler manifold with nonnegative bisectional curvature, maximal volume growth and faster than quadratic scalar curvature decay, then M^n is isometrically biholomorphic to $\mathbb{C}^n$. Later, Mok [**26**] proved that if M^n has positive bisectional curvature, maximal volume growth and quadratic scalar curvature decay, then M is an affine algebraic variety. As a consequence, if $n = 2$ and the sectional curvature is positive,

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then M^n is biholomorphic to $\mathbb{C}^2$ by a result of Ramanujan [**35**]. In this case, dimension 2, it is known that the condition on the sectional curvature can be relaxed and the decay of the scalar curvature can also be removed, see [**11**, **13**, **30**]. In higher dimensions and in general, the conjecture is still very open, and until now, this has been so even if M^n is assumed to have bounded curvature and maximal volume growth. In this paper (Corollary 1.1) we show that the conjecture is true in all dimensions provided M^n has bounded curvature and maximal volume growth.

In his thesis [**38**], Shi used the following Ricci flow of Hamilton [**20**] to better understand the uniformization conjecture in the case of (M^n, g) as in Mok's paper [**26**]:

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{g}_{i\bar{j}}(x,t) &= -\tilde{R}_{i\bar{j}}(x,t) \\ \tilde{g}_{i\bar{j}}(x,0) &= \tilde{g}_{i\bar{j}}(x). \end{aligned} \tag{1.1}$$

On a Kähler manifold, (1.1) is referred to as the Kähler-Ricci flow. In [**38**, **37**], Shi obtained several important results for this flow including short time existence for general solutions, and long time existence together with many useful estimates in the above case; see Theorem 2.1 for more details. Although the results in [**38**] did not actually prove uniformization in this case[1] , their importance remains fundamental to the study of Yau's Conjecture; in particular, in the above mentioned works [**11**], [**13**], [**30**] as well as the present paper.

In this paper, by studying the asymptotic behavior of the Kähler-Ricci flow (1.1) in more detail, we will prove the following uniformization theorem:

Theorem 1.1. *Let $(M^n, \widetilde{g})$ be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Let R be the scalar curvature of M. Suppose*

(i) $\mathrm{Vol}\,(B(p,r)) \geq C_1 r^{2n}; \quad \forall r \in [0,\infty)$ *for some* $p \in M$,
(ii) $\frac{1}{V_x(r)}\int_{B_x(r)} R \leq \frac{C_2}{1+r^2}$ *for all* $x \in M$ *and for all* $r > 0$,

for some positive constants C_1, C_2. Then M is biholomorphic to $\mathbb{C}^n$. Moreover, condition (i) *can be removed if M has positive curvature operator.*

In [**43**], Yau conjectured that (i) actually implies (ii). This has recently been confirmed by Chen-Tang-Zhu [**11**] for the case of dimension 2, Chen-Zhu [**13**] for higher dimensions under the additional assumption of nonnegative curvature operator and recently by Ni [**30**] for all dimensions. Hence we have:

[1] This was observed later on in [**10**]. Also see [**7**].

Corollary 1.1. *Let $(M^n, \widetilde{g})$ be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature and maximal volume growth. Then M is biholomorphic to $\mathbb{C}^n$.*

Also, under only assumption (ii) in Theorem 1.1, and assuming the curvature operator is nonnegative, one can prove that the universal cover of M is biholomorphic to $\mathbb{C}^n$.

In order to prove Theorem 1.1, we first obtain some results on the long time behavior of the Kähler-Ricci flow (1.1) which may be of independent interest. For these, it will be more convenient to consider the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} g(t) = -Rc(t) - g(t) \tag{1.2}$$

where $g(t) = e^{-t}\widetilde{g}(e^t)$ (for $\widetilde{g}(t)$ as in (1.1)) and $Rc(t)$ is the Ricci curvature of $g(t)$. Under the assumptions of Theorem 1.1 we have:

Theorem 1.2. *Let $(M^n, \widetilde{g})$ be as in Theorem 1.1 with either maximal volume growth or positive curvature operator, and let $g(x,t)$ be as in (1.2). Let $p \in M$ be any point. Then the eigenvalues of $Rc(p,t)$ with respect to $g(p,t)$ will converge as $t \to \infty$. Moreover, if $\mu_1 > \mu_2 > \cdots > \mu_l$ are the distinct limits of the eigenvalues, then $V = T_p^{(1,0)}(M^n)$ can be decomposed orthogonally with respect to $g(0)$ as $V_1 \oplus \cdots \oplus V_l$ so that the following are true:*

(i) *If v is a nonzero vector in V_i for some $1 \le i \le l$, and let $v(t) = v/|v|_{g(t)}$, then*
$$\lim_{t\to\infty} Rc(v(t), \bar{v}(t)) = \mu_i$$
and thus
$$\lim_{t\to\infty} \frac{1}{t} \log \frac{|v|^2_{g(t)}}{|v|^2_{g(0)}} = -\mu_i - 1.$$
Moreover, both convergences are uniform over all $v \in V_i \setminus \{0\}$.

(ii) *For $1 \le i, j \le l$ and for nonzero vectors $v \in V_i$ and $w \in V_j$ where $i \ne j$, $\lim_{t\to\infty}\langle v(t), w(t)\rangle_t = 0$ and the convergence is uniform over all such nonzero vectors v, w.*

(iii) $\dim_{\mathbb{C}}(V_i) = n_i - n_{i-1}$ *for each i (see §4 for definition of n_i).*

(iv)
$$\sum_{i=1}^{l} (-\mu_i - 1)\dim_{\mathbb{C}} V_i = \lim_{t\to\infty} \frac{1}{t} \log \frac{\det(g_{i\bar{j}}(t))}{\det(g_{i\bar{j}}(0))}.$$

In terms of the Kähler-Ricci flow, the theorem says that $(M^n, g(t))$ asymptotically behaves like a gradient Kähler-Ricci soliton of expanding type at p; see Proposition 3.2 for more details. We remark that the first example of gradient expanding Kähler-Ricci soliton was constructed by Cao [**6**]. Also, conclusions (i) and (ii) basically say that $Rc(p,t)$ can be 'simultaneously diagonalized' near $t = \infty$ in some sense. From the point of view of dynamical systems, conclusions (i), (iii) and (iv) together basically say that $g(t)$ is Lyapunov regular; see [**1**].

A main theme in the proof of Theorem 1.1 is the connection between the Kähler-Ricci flow and a certain class of dynamical systems. This can be sketched as follows. By Theorem 2.1 in next section, we can construct a biholomorphism from each element in a sequence of open sets exhausting M onto a fixed ball in $\mathbb{C}^n$. By sequentially identifying these open sets, the results in Theorem 1.2 can be interpreted in terms of the dynamics of a randomly iterated sequence of biholomorphisms as in [**23**]. Using the results of Theorem 1.2 in this setting, and using techniques developed by Rosay-Rudin [**36**] and Jonsson-Varolin [**23**], we then proceed to assemble these biholomorphisms into a global biholomorphism from M to $\mathbb{C}^n$.

The paper is organized as follows. In §2 we review the main results on the Kähler-Ricci flow (2.1) which we use later. In §3 and §4 we study the asymptotic behavior of the Kähler-Ricci flow on M as $t \to \infty$. The focus of §3 will primarily be on the global asymptotics of the Kähler-Ricci flow on M while that of §4 will be purely local. We believe that these asymptotics should be of independent interest to the study of the Kähler-Ricci flow. Finally, in §5 we will prove Theorem 1.1 and its corollaries.

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2. The Kähler Ricci flow

In this section we will collect some known results on Kähler-Ricci flow which will be used in this work. Recall that on a complete noncompact Kähler manifold $(M^n, \tilde{g}_{i\bar{j}}(x))$, the Kähler-Ricci flow equation is:

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{g}_{i\bar{j}}(x,t) &= -\tilde{R}_{i\bar{j}}(x,t) \\ \tilde{g}_{i\bar{j}}(x,0) &= \tilde{g}_{i\bar{j}}(x). \end{aligned} \tag{2.1}$$

Theorem 2.1. *Let $(M^n, \tilde{g})$ be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Suppose there is a constant $C > 0$ such that its scalar curvature $\tilde{R}$ satisfies*

$$\frac{1}{V_x(r)}\int_{B_x(r)} \tilde{R}\, dV_g \le \frac{C}{1+r^2} \tag{2.2}$$

for all $x \in M$ and for all $r > 0$. Then the Kähler-Ricci flow (2.1) *has a long time solution $\tilde{g}_{\alpha\bar{\beta}}(x,t)$ on $M \times [0,\infty)$. Moreover, the following are true:*

(i) *For any $t \ge 0$, $\tilde{g}(x,t)$ is Kähler with nonnegative holomorphic bisectional curvature.*

(ii) *For any integer $m \geq 0$, there is a constant C_1 depending only on m and the initial metric such that*

$$\|\nabla^m \tilde{R}m\|^2(x,t) \leq \frac{C_1}{t^{2+m}},$$

for all $x \in M$ and for all $t \geq 0$, where ∇ is the covariant derivative with respect to $\tilde{g}(t)$ and the norm is also taken in $\tilde{g}(t)$.

(iii) *If in addition $(M, \widetilde{g}(0))$ has either maximum volume growth or positive curvature operator, then there exists a constant $C_2 > 0$ depending only on the initial metric such that the injectivity radius of $\tilde{g}(t)$ is bounded below by $C_2 t^{1/2}$ for all $t \geq 1$.*

Proof. (i) and (ii) are mainly obtained by Shi [**37**, **38**, **40**] (also see [**33**]). To prove (iii), suppose $\widetilde{g}(0)$ has positive curvature operator. Then by [**22**] we know that positive curvature operator is preserved under (2.1), and thus $g(t)$ has positive sectional curvature at every time t. From this and the estimates in (ii), we can conclude by the results in [**19**] that (iii) is true in the case of positive curvature operator. (See [**14**, p. 14] for a description of how to prove this.) In the case of maximal volume growth, (iii) has been observed in [**10**]. In fact, if $Vol_0(B_x(r)) \geq Cr^{2n}$ for some $C > 0$ for the initial metric, then we also have $\mathrm{Vol}_t(B_x(r)) \geq Cr^{2n}$ for the metric $g(t)$ for all $t \geq 0$ with the same constant C, see [**10**] for example. Combining this with the curvature estimates (ii) and the injectivity radius estimates in [**9**], (iii) follows in this case. q.e.d.

We now consider the following normalization of (2.1):

$$\frac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t) - g_{i\bar{j}}(x,t). \tag{2.3}$$

It is easy to verify that if $\tilde{g}(x,t)$ solves (2.1), then

$$g(x,t) = e^{-t}\tilde{g}(x, e^t) \tag{2.4}$$

is a solution to (2.3). Thus for $\tilde{g}(x,t)$ as in Theorem 2.1, $g(x,t)$ in (2.4) is defined for $-\infty < t < \infty$. Note that $\lim_{t\to-\infty} g(x,t) = \widetilde{g}(x)$ which is the initial data of (2.1). The results in Theorem 2.1 can be translated to the following results for a solution to (2.3):

Corollary 2.1. *Let $\tilde{g}(x,t)$ be as in Theorem 2.1 and let $g(x,t)$ be given by (2.4). Then the following are true:*

(i) *For any $-\infty < t < \infty$, $g(x,t)$ is Kähler with nonnegative holomorphic bisectional curvature.*

(ii) *For any integer $m \geq 0$, there is a constant C_1 depending only on m and the initial metric such that*

$$\|\nabla^m Rm\|^2(x,t) \leq C_1,$$

for all $x \in M$ and for all $t \geq 0$, where ∇ is the covariant derivative with respect to $g(t)$ and the norm is also taken in $g(t)$.

(iii) *If in addition* $(M, \widetilde{g}(0))$ *has either maximum volume growth or positive curvature operator, then there exists a constant* $C_2 > 0$ *depending only on the initial metric such that the injectivity radius of* $g(t)$ *is bounded below by* C_2 *for all* $t \geq 0$.

We shall need the following.

Proposition 2.1. *Let* (M^n, g) *be a complete Kähler manifold with nonnegative holomorphic bisectional curvature such that* $|Rm| + |\nabla Rm| \leq C_1$ *and the injectivity of* M *is larger than* r_0. *Then there exist positive constants* r_1, r_2 *and* C_2 *depending only on* C_1, r_0 *and* n *such that for each* $p \in M$, *there is a holomorphic map* Φ *from the Euclidean ball* $\widehat{B}_0(r_1)$ *at the origin of* $\mathbb{C}^n$ *to* M *satisfying the following:*

(i) Φ *is a biholomorphism from* $\widehat{B}_0(r_1)$ *onto its image;*
(ii) $\Phi(0) = p$*;*
(iii) $\Phi^*(g)(0) = g_\epsilon$*;*
(iv) $\frac{1}{r_2} g_\epsilon \leq \Phi^*(g) \leq r_2 g_\epsilon$ *in* $\widehat{B}(0, r_1)$.

where g_ϵ *is the standard metric on* $\mathbb{C}^n$.

Proof. This is in fact a special case of Proposition 1.2 in [**42**], see also [**40**, **10**]. For the sake of completeness, we sketch the proof as follows.

By the assumption on the injectivity radius, let $x_1, \ldots, x_{2n}$ be normal coordinates on $B_p(r_0)$ so that if $z_i = x_i + \sqrt{-1}x_{n+i}$ are standard complex coordinates of $\mathbb{C}^n$, then $\frac{\partial}{\partial z_i}$ form a basis for $T_p^{(1,0)}(M)$ at p. Hence there is a diffeomorphism F from $B_p(r_0)$ onto $\widehat{B}_0(r_0)$ such that $F(p) = 0$ and $dF \circ J = \widehat{J} \circ dF$ at 0 where $\widehat{J}$ is the standard complex structure on $\mathbb{C}^n$ and J is the complex structure of M. By [**21**], the components of the metric g with respect to coordinates x_i satisfies

$$|\delta_{ij} - g_{ij}| \leq C_2 |x|^2, \quad \frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2\delta_{ij},$$

$$\left| \frac{\partial^2}{\partial x_k \partial x_l} g_{ij} \right| \leq C_2$$

and

$$\left| \frac{\partial}{\partial x_k} g_{ij} \right| (x) \leq C_2 |x|$$

in $B_p(r_1)$ for some positive constants r_1, C_2 depending only on C_1, r_0 and n. Here $|x|^2 = \sum_i (x_i)^2$. In the following C_i's and r_i's always denote positive constants depending only on C_1, r_0 and n. Hence if r_1 small enough, $\sqrt{-1}\partial\bar{\partial} \log \rho^2 \geq -C_3 \omega$ and the eigenvalues of the Hessian of ρ^2 are bounded below by C_4. Here ρ is the distance from p and ω is the Kähler form. One can prove that $|J - \widehat{J}| \leq C_5 \rho^2$, where we also denote the pull back of $\widehat{J}$ under F with $\widehat{J}$, see [**10**]. The i-th component

$z_i = x_i + \sqrt{-1}x_{n+i}$ of the map F when considered as a map from $B_p(r_0)$ to $\mathbb{C}^n$ satisfies

$$|\overline{\partial} z_i| \le C_6\rho^2. \tag{2.5}$$

As in [**29**], by Corollary 5.3 in [**15**], using the weight function $\varphi = (n+2)\log\rho^2 + C_7\rho^2$ for some C_7 so that $\sqrt{-1}\partial\overline{\partial}\varphi \ge C_8\omega$, one can solve $\overline{\partial} u_i = \overline{\partial} z_i$ in $B_p(r_1)$ with

$$\int_{B_p(r_1)} |u_i|^2 e^{-\varphi} \le \frac{1}{C_8}\int_{B_p(r_1)} |\overline{\partial} z_i|^2 e^{-\varphi} \le C_9 \tag{2.6}$$

for some C_9. Here we have used the fact that $Ric \ge 0$ and (2.5). From this, it is easy to see that $u_i(p) = 0$ and $du_i(p) = 0$. Moreover, from the fact that $z_i - u_i$ is holomorphic one can prove that on $B_p(r_1/2)$,

$$|u_i| + |\nabla u_i| + |\nabla^2 u_i| \le C_{10}$$

by (2.6), mean value inequality in [**25**], gradient estimates and Schauder estimates. Hence we have $|\nabla u_i| \le C_{11}\rho$ and $|u_i| \le C_{11}\rho^2$. So the map Φ given by $\Phi^{-1} = (z_1 - u_1, \dots, z_n - u_n)$ will satisfy the conditions in the proposition if r_1 is small enough and r_2 is large enough. q.e.d.

Using this and Corollary 2.1, we have the following (also see [**42**, **40**]).

Corollary 2.2. *Let $(M^n, \widetilde{g}(0))$ and $g(x,t)$ be as in Corollary 2.1 such that $(M, \widetilde{g}(0))$ has either maximum volume growth or positive curvature operator. Let $p \in M$ be a fixed point. Then there are constants r_1 and r_2 depending only on the initial metric such that for every $t > 0$ there exists a holomorphic map $\Phi_t : \widehat{B}_0(r_1) \subset \mathbb{C}^n \to M$ satisfying:*

(i) *Φ_t is a biholomorphism from $\widehat{B}_0(r_1)$ onto its image;*
(ii) *$\Phi_t(0) = p$;*
(iii) *$\Phi_t^*(g(t))(0) = g_\epsilon$;*
(iv) *$\frac{1}{r_2}g_\epsilon \le \Phi_t^*(g(t)) \le r_2 g_\epsilon$ in $\widehat{B}_0(r_1)$;*

where g_ϵ is the standard metric on $\mathbb{C}^n$, and $\widehat{B}_0(r_1)$ is the Euclidean ball of radius r_1 with center at the origin in $\mathbb{C}^n$. Moreover, the following are true:

(v) *For any $t_k \to \infty$ and for any $0 < r < r_1$, the family $\{\Phi_{t_k}(\widehat{B}_0(r))\}_{k\ge 1}$ exhausts M and hence M is simply connected.*
(vi) *If T is large enough, then $F_{i+1} = \Phi^{-1}_{(i+1)T} \circ \Phi_{iT}$ maps $\widehat{B}_0(r_1)$ into $\widehat{B}_0(r_1)$ for each i, and there is $0 < \delta < 1$, $0 < a < b < 1$ such that*

$$|F_{i+1}(z)| \le \delta|z|$$

for all $z \in \widehat{B}_0(r_1)$, and

$$a|v| \le |F'_{i+1}(0)(v)| \le b|v|$$

for all v for all i.

Proof. (i)–(iv) follows immediately from Proposition 2.1 and Corollary 2.1. To prove (v), observe that $B_p^t(r/r_2) \subset \Phi_t(\widehat{B}_0(r))$ by (i) and (iv), where $B_p^t(R)$ is the geodesic ball of radius R with respect to $g(t)$ with center at p. On the other hand, by (2.3), $|v|^2_{g(t)} \le e^{-t}|v|^2_{g(0)}$ and so $B_p^0(R) \subset B_p^t(e^{-t/2}R)$. From this it is easy to see that (v) is true.

To prove (vi), let v be a $(1,0)$ vector on M and denote $|v|_t$ to be the length of v with respect to $g(t)$. By (2.3) and Corollary 2.1

$$\begin{aligned}
-|v|_t^2 &\ge \frac{d}{dt}|v|_t^2 \\
&= -Rc_{\widetilde{g}}(v,v) - \widetilde{g}(v,v) \\
&\ge -C_1\widetilde{g}(v,v) - \widetilde{g}(v,v) \\
&\ge -(C_1+1)|v|_t^2
\end{aligned} \tag{2.7}$$

for some constant $C_1 > 0$ which is independent of v and t. Hence for any $T > 0$ and $i \ge 1$,

$$e^{-T} \ge \frac{|v|^2_{(i+1)T}}{|v|^2_{iT}} \ge e^{-(C_1+1)T}. \tag{2.8}$$

Since

$$\Phi_{iT}(\widehat{B}_0(r_1)) \subset B_p^{iT}(r_2r_1) \subset B_p^{(i+1)T}(e^{-T/2}r_2r_1),$$

and $\Phi_{(i+1)T}(\widehat{B}_0(r_1)) \supset B_p^{(i+1)T}(r_1/r_2)$, it follows that F_{i+1} is defined on $\widehat{B}_0(r_1)$ and $F_{i+1}(\widehat{B}_0(r_1)) \subset \widehat{B}_0(r_1)$ if T is large enough. From (iv) and (2.8), it is easy to see that there is $0 < \delta < 1$, such that

$$|F_{i+1}(z)| \le \delta|z|$$

for all $z \in \widehat{B}_0(r_1)$ for all i if T is large. From (ii), (iii) and (2.8), we can also find $0 < a < b < 1$ such that

$$a|v| \le |F'_{i+1}(0)(v)| \le b|v|$$

for all v and for all i. This completes the proof of the corollary. q.e.d.

In §5, we will use the maps Φ_t to construct a biholomorphism from M to $\mathbb{C}^n$.

3. Asymptotic behavior of Kähler Ricci flow (I)

Let $(M^n, \tilde{g}_{i\bar{j}}(x))$ be as in Theorem 2.1 satisfying (2.2). Let $\widetilde{g}(x,t)$ and $g(x,t)$ be the corresponding solutions to (2.1) and (2.3) respectively. Then for any point $p \in M$, we will show that the eigenvalues of $Rc(p,t)$ relative to $g(p,t)$ actually converge to a fixed set of numbers as $t \to \infty$. Here $Rc(p,t)$ is the Ricci tensor of $g(t)$ at p. If in addition $(M, \widetilde{g})$ has maximal volume growth with positive Ricci curvature or has positive curvature operator, then we will show that for any $p \in M$, $(M, g(x,t), p)$

approaches an expanding gradient Kähler-Ricci soliton as $t \to \infty$ in the sense of limiting solutions to the Kähler-Ricci flow ([**21**]).

Proposition 3.1. *Let $(M^n, g_{i\bar{j}}(x))$, $\widetilde{g}(x,t)$, $g(x,t)$ be as in Theorem 2.1 satisfying (2.2). Let $p \in M$ be a fixed point in M and let $\lambda_1(t) \geq \cdots \geq \lambda_n(t) \geq 0$ be the eigenvalues of $R_{i\bar{j}}(p,t)$ relative to $g_{i\bar{j}}(p,t)$.*

(i) *For any $\tau > 0$,*
$$\frac{\det(R_{i\bar{j}}(p,t) + \tau\delta_{ij})}{\det(g_{i\bar{j}}(p,t))}$$
is nondecreasing in t.

(ii) *Assume in addition that $\tilde{g}_{i\bar{j}}(x)$ has positive Ricci curvature. Then there is a constant $C > 0$ such that $\lambda_n(t) \geq C$ for all t.*

(iii) *For $1 \leq i \leq n$ the limit $\lim_{t\to\infty} \lambda_i(t)$ exists.*

(iv) *Let $\mu_1 > \cdots > \mu_l \geq 0$ be the distinct limits in (iii) and let $\rho > 0$ be such that $[\mu_k - \rho, \mu_k + \rho]$, $1 \leq k \leq l$ are disjoint. For any t, let $E_k(t)$ be the sum of the eigenspaces corresponding to the eigenvalues $\lambda_i(t)$ such that $\lambda_i(t) \in (\mu_k - \rho, \mu_k + \rho)$. Let $P_k(t)$ be the orthogonal projection (with respect to $g(t)$) onto $E_k(t)$. Then there exists $T > 0$ such that if $t > T$ and if $w \in T_p^{(1,0)}(M)$, $|P_k(t)(w)|_t$ is continuous in t, where $|\cdot|_t$ is the length measured with respect to the metric $g(p,t)$.*

Proof.

(i): By the Li-Yau-Hamilton (LYH) inequality in [**3**] and in [**4**, Theorem 2.1], if

$$Z_{i\bar{j}} = \frac{\partial R_{i\bar{j}}}{\partial t} + g^{k\bar{l}} R_{i\bar{l}} R_{k\bar{j}} + R_{i\bar{j}} \tag{3.1}$$

then

$$Z_{i\bar{j}} w^i w^{\bar{j}} \geq 0 \tag{3.2}$$

for any $w \in T^{(1,0)}(M)$. For any $\tau > 0$, denote
$$\phi(t) = \frac{\det(R_{i\bar{j}} + \tau g_{i\bar{j}})}{\det(g_{i\bar{j}})}$$
at (p,t). Denote $p_{i\bar{j}} = R_{i\bar{j}} + \tau g_{i\bar{j}}$ as in [**3**] and note that $(p_{i\bar{j}})$ is invertible and denote its inverse by $(p^{i\bar{j}})$. We have

(3.3)
$$\begin{aligned}
\frac{\partial}{\partial t}\log\phi &= p^{i\bar{j}}\frac{\partial}{\partial t}p_{i\bar{j}} - g^{i\bar{j}}\frac{\partial}{\partial t}g_{i\bar{j}} \\
&= p^{i\bar{j}}\left(\frac{\partial}{\partial t}R_{i\bar{j}} - \tau(R_{i\bar{j}} + g_{i\bar{j}})\right) + g^{i\bar{j}}\left(R_{i\bar{j}} + g_{i\bar{j}}\right) \\
&\geq p^{i\bar{j}}\left(-g^{k\bar{l}}R_{i\bar{l}}R_{k\bar{j}} - R_{i\bar{j}} - \tau(R_{i\bar{j}} + g_{i\bar{j}})\right) + g^{i\bar{j}}\left(R_{i\bar{j}} + g_{i\bar{j}}\right) \\
&= p^{i\bar{j}}\left(-g^{k\bar{l}}R_{i\bar{l}}R_{k\bar{j}} - (\tau+1)p_{i\bar{j}}\right) + \tau^2 p^{i\bar{j}}g_{i\bar{j}} + g^{i\bar{j}}(R_{i\bar{j}} + g_{i\bar{j}})
\end{aligned}$$

where we have used (3.1) and (3.2). Now at the point (p,t), we choose a unitary basis such that $g_{i\bar{j}} = \delta_{ij}$ and $R_{i\bar{j}} = \lambda_i \delta_{ij}$. Then $p_{i\bar{j}} = (\lambda_i + \tau)\delta_{ij}$ and $p^{i\bar{j}} = (\lambda_i + \tau)^{-1}\delta_{ij}$. Hence we have

$$\begin{aligned} \frac{\partial}{\partial t}\log\phi &\geq -\sum_{i=1}^{n}\frac{\lambda_i^2}{\lambda_i+\tau} - (\tau+1)n + \sum_{i=1}^{n}\frac{\tau^2}{\lambda_i+\tau} + \sum_{i=1}^{n}\lambda_i + n \\ &= \sum_{i=1}^{n}\left(\frac{-\lambda_i^2}{\lambda_i+\tau} - \tau + \frac{\tau^2}{\lambda_i+\tau} + \lambda_i\right) \\ &= 0. \end{aligned} \tag{3.4}$$

From this (i) follows.

(ii): By (i), we conclude that $\frac{\det(R_{i\bar{j}}(p,t))}{\det(g_{i\bar{j}}(p,t))}$ is nondecreasing (this fact has been proved in [**3**]). Moreover,

$$\lim_{t\to-\infty}\frac{\det(R_{i\bar{j}}(p,t))}{\det(g_{i\bar{j}}(p,t))} = \frac{\det(R_{i\bar{j}}(p))}{\det(g_{i\bar{j}}(p))}$$

where the right side is in terms of the initial metric g for (2.1). Since the Ricci curvature is assumed to be positive, $\frac{\det(R_{i\bar{j}}(p,t)}{\det(g_{i\bar{j}}(p,t))} \geq C_1$ for some positive constant C_1 for all t. On the other hand, by Corollary 2.1 there is a constant C_2 independent of t such that $\lambda_1(t) \leq C_2$. From these two facts, part (ii) of the proposition follows.

(iii): Choose a unitary basis $v_1, \ldots, v_n$ for $T_p^{(1,0)}(M)$ with respect to the metric $g(p,0)$. Using the Gram-Schmidt process, we can obtain a unitary basis $v_1(t), \ldots, v_n(t)$ for $g(p,t)$. Since $g(t)$ is smooth in t, we conclude that the $v_i(t)$'s are smooth in t. That is to say, $v_i(t)$ is a linear combination of a fixed basis of $T_p^{(1,0)}(M)$ with smooth coefficients. Denote by $R_{i\bar{j}}(t) = Rc(v_i(t), \bar{v}_j(t))$ the components of $Rc(p,t)$ with respect to this basis. Then $R_{i\bar{j}}(t)$ is also smooth in t. By (i) and Corollary 2.1, for any $\tau > 0$,

$$\lim_{t\to\infty}\det(R_{i\bar{j}}(t) + \tau\delta_{ij}) = c(\tau) \tag{3.5}$$

exists.

Now $\lambda_i(t)$ are uniformly bounded functions in t. To prove (iii), it is sufficient to prove that if $t_k \to \infty$, $t'_k \to \infty$ and

$$\lim_{k\to\infty}\lambda_i(t_k) = \tau_i, \ \lim_{k\to\infty}\lambda_i(t'_k) = \tau'_i$$

for all i, then $\tau_i = \tau'_i$.

By (3.5), we have

$$\prod_{i=1}^{n}(\tau_i + \tau) = \prod_{i=1}^{n}(\tau'_i + \tau)$$

for all $\tau > 0$. Since $\tau_1 \geq \cdots \geq \tau_n$ and $\tau'_1 \geq \cdots \geq \tau'_n$, we must have $\tau_i = \tau'_i$. This completes the proof of (iii).

(iv): By (iii), if T is large enough, for each i we have $\lambda_i(t) \in (\mu_k - \rho, \mu_k + \rho)$ for some k for all $t \geq T$. Hence $\dim E_k(t)$ is constant in t for $t \geq T$. Let $P_k(t)$ be the orthogonal projection (with respect to $g(t)$) onto $E_k(t)$. We also denote the matrix of this projection, with respect to the basis $v_1(t), \dots, v_n(t)$ in (iii), by $P_k(t)$. Then

$$P_k(t) = -\frac{1}{2\pi\sqrt{-1}} \int_C (R_{i\bar{j}} - z\delta_{ij})^{-1} dz.$$

where C is a circle on the complex plane with center at μ_k and radius ρ, see [**24**, p. 40] for example. It is easy to see that the matrix valued function $P_k(t)$ is continuous in t. Hence (iv) is true. q.e.d.

Remark 1. The facts that the scalar curvature $R(t)$ and $\det(R_{i\bar{j}}(t))/\det(g_{i\bar{j}}(t))$ are nondecreasing have been proved in [**3**]

Next, we will study the global asymptotic behavior of the manifolds $(M^n, g(t))$ as $t \to \infty$. We will need the following lemma from [**16**]:

Lemma 3.1. *Let $(M^n, g_{i\bar{j}})$ be a complete noncompact Kähler manifold with bounded curvature. Suppose there is a smooth function f such that $\sqrt{-1}\partial\bar{\partial} f = Rc$. Let $g_{i\bar{j}}(t)$ and $\widehat{g}_{i\bar{j}}(t)$ be two solutions of* (2.1) *on $M \times [0, T]$, $T > 0$ with the same initial data $g_{i\bar{j}}$ such that*

$$c^{-1} g_{i\bar{j}}(x) \leq g_{i\bar{j}}(x,t), \widehat{g}_{i\bar{j}}(x,t) \leq c g_{i\bar{j}}(x) \tag{3.6}$$

for some constant $c > 0$ for all $(x,t) \in M \times [0,T]$. Then $g_{i\bar{j}}(x,t) = \widehat{g}_{i\bar{j}}(x,t)$ on $M \times [0,T]$.

In [**4**] it was proved by Cao that for any $t_k \to \infty$, if $|R(p_k, t_k)|$ is the maximum of the scalar curvature on M at t_k, then the blow down limit of $g(t)$ along (p_k, t_k) is an expanding gradient Kähler-Ricci soliton. Recently, it is shown by Ni in [**31**] that the result is still true for an arbitrary sequence $p_k \in M$, $t_k \to \infty$. In the special case that the sequence $p_k = p$ is fixed at an arbitrary $p \in M$, the result follows from a rather simple observation and the argument in [**4**], which we present below.

Proposition 3.2. *Assume the conditions and notation of Proposition* 3.1. *In addition, assume the initial metric $\widetilde{g}(x,0) = \tilde{g}_{i\bar{j}}(x)$ of* (2.1) *has either maximal volume growth with positive Ricci curvature or has positive curvature operator. Let $p \in M$ be a fixed point. The given any $t_k \to \infty$, we can find a subsequence also denoted by t_k, a complete noncompact complex manifold N^n, and a family of Kähler metrics $h(t)$ on N satisfying* (2.3) *for all $t \in \mathbb{R}$ such that $(M^n, g_k(t))$, where $g_k(t) = g(t_k + t)$ for all $t \in \mathbb{R}$, converges to $(N, h(t))$ in the following sense: There exists a family of diffeomorphisms $F_k : U_k \subset N \to M$ with the following properties.*

(i) *Each* U_k *contains* o *where* $o \in N$ *is a fixed point and* $F_k(o) = p$.
(ii) U_k *is open and the* U_k*'s exhaust* N.
(iii) $(U_k, F_k^*(g_k(t)))$ *converges in* C^∞ *norm uniformly on compact sets to* $h(t)$ *in* $N \times \mathbb{R}$.

Moreover $(N, h(t))$ *is a gradient Kähler-Ricci soliton. More precisely, there is a family of biholomorphisms* ϕ_t *of* N *determined by the gradient of some real valued function such that* o *is a fixed point of each* ϕ_t *and* $\phi_t^*(h(0)) = h(t)$ *for all* $t \geq 0$.

Proof. The existence of t_k, N, $h(t)$ and F_k satisfying (i)–(iii) is a consequence of Theorem 2.1 and the compactness theorem of Hamilton [**21**].

We now prove the last assertion in the Proposition. Begin by noting that $\lim_{t\to\infty} R(t)$ exists by Proposition 3.1, where $R(t)$ is the scalar curvature of $g(t)$ at p. Let $R^h(t)$ be the scalar curvature of $h(t)$ at o. Then for any t, t'

$$R^h(t) = \lim_{k\to\infty} R(t_k + t) = \lim_{k\to\infty} R(t_k + t') = R^h(t'). \tag{3.7}$$

Now consider the metric $\widetilde{h}(t) = th(\log t)$ for $t \geq 1$. Then $\widetilde{h}$ is a solution to (2.1) on $N \times [1, \infty)$. Also, since $g(t)$ has uniformly bounded curvature in spacetime by Corollary 2.1, $h(t)$ also has uniformly bounded curvature in spacetime. By Proposition 3.1 (ii), the Ricci curvature of $h(t)$ at p is positive. Moreover, by Theorem 2.1, the facts that M is simply connected and that the metrics $g(t)$ are decreasing in t, we can conclude that N is simply connected. By [**5**], it is easy to see that $h(t)$ and hence $\widetilde{h}(t)$ have positive Ricci curvature. Now (3.7) implies that $t\widetilde{R}(t)$ is constant where $\widetilde{R}(t)$ is the scalar curvature of $\widetilde{h}(t)$ at p. Hence $\frac{\partial}{\partial t}(t\widetilde{R}) = 0$ for all t, and by the proof of Theorem 4.2 in [**4**], there is a real valued function f such that $f_{i\bar{j}}(x) = \widetilde{R}_{i\bar{j}}(x, 1) + \widetilde{h}_{i\bar{j}}(x, 1)$ on N with $f_{ij} \equiv 0$ and $\nabla f(o) = 0$.

Let $\phi_t(x)$ be the integral curve of $-\frac{1}{2}\nabla f$ on N with initial point x. We claim that $\phi_t(x)$ is defined for all x and t. Let $\widetilde{h}_{AB}$ and $\widetilde{R}_{AB}$ be the Riemannian metric $2Re(\widetilde{h}_{i\bar{j}})$ and Ricci curvature of $\widetilde{h}_{AB}$. Then $f_{AB} = \widetilde{R}_{AB} + \widetilde{h}_{AB}$. Observe that as in ([**22**, §20]), we have

$$|\nabla f|^2 + \widetilde{R} = 2f + 2C_1 \tag{3.8}$$

where $\widetilde{R}$ is the scalar curvature of $\widetilde{h}(1)$ and C_1 is a constant.

Now, as long as $\phi_t(x)$ is defined in on $[-T, 0]$ for $T > 0$, then for $0 \le t \le T$

$$
\begin{aligned}
(3.9) \qquad f(\phi_{-t}(x)) - f(x) &= \int_0^{-t} \frac{d}{ds} f(\phi_s(x)) ds \\
&= \int_0^{-t} \left\langle \nabla f(\phi_s(x)), \frac{d}{ds}\phi_s(x) \right\rangle ds \\
&= \frac{1}{2} \int_0^{t} |\nabla f(\phi_{-s}(x))|^2 ds \\
&\le \int_0^{t} f(\phi_{-s}(x)) ds + C_1 t
\end{aligned}
$$

by (3.8). Hence we have $f(\phi_{-t}(x)) \le C_2$ for some constant depending only on T, C_1 and $f(x)$. One can also prove that $f(\phi_t(x)) \le f(x)$ for $t > 0$ as long as $\phi_t(x)$ is defined up to t. Since f is an exhaustion function by ([**7**], Lemma 3.1), we conclude that $f(\phi_t(x))$ remains in a fixed compact set on any bounded interval of $\mathbb{R}$ as long as ϕ_t is defined on that interval. From this it is easy to see that $\phi_t(x)$ is defined for all t. Since ∇f is a holomorphic vector field, ϕ_t is in fact a biholomorphism on N for all t.

Let $h_1(t) = \phi_t^*(\widetilde{h}(1)) = \phi_t^*(h(0))$ and let $\widetilde{h}_1(t) = t h_1(\log t)$ for $t \ge 1$. We will show that $\widetilde{h}_1(t) = \widetilde{h}(t)$ for $t \ge 1$. Since $h(t)$ has nonnegative holomorphic bisectional curvature such that its scalar curvature is uniformly bounded in spacetime, $\widetilde{h}(t)$ also has nonnegative holomorphic bisectional curvature with $t\widetilde{R}(t)$ being uniformly bounded in spacetime where $\widetilde{R}(t)$ is the scalar curvature of $\widetilde{h}(t)$. By [**33**, Theorem 2.1] and [**32**, Theorem 5.1], we can find a potential function for the Ricci tensor of $\widetilde{h}(1)$. Since the curvature of $\widetilde{h}$ and $\widetilde{h}_1$ are uniformly bounded on $M \times [0, T]$ for fixed $T > 0$, it is easy to see that they satisfy (3.6). By Lemma 3.1, we conclude that $\widetilde{h}_1(t) = \widetilde{h}(t)$ for $t \ge 1$. Hence $h_1(t) = h(t)$ for all $t \ge 0$. This completes the proof of the proposition. q.e.d.

Let $t_k \to \infty$ such that $(M, g_k(t))$ converges to $(N, h(t))$ as in Proposition 3.2. We will describe this convergence in terms of the convergence of certain specific quantities. For simplicity, we identify $(M, g_k(t))$ near p with $(U, F_k^*(g_k(t))$ for some open set $U \subset N$ containing o. Let J_k be the complex structure on U given by the pullback of the complex structure of M under F_k and let J be the complex structure of N. By taking a subsequence we may also assume that $J_k \to J$. Let $w_k \in T_p^{(1,0)}(M)$ with $|w_k|_{g_k(0)} = 1$ and let $w_k(t) = w_k/|w_k|_{g_k(t)}$ for $t \ge 0$. Denote $w_k = x_k - \sqrt{-1} J_k(x_k)$ where x_k is in the real tangent space of M at p which is identified with the real tangent space of N at o. Assume that $x_k \to x$. Then $J_k(x_k) \to J(x)$. Let $u = x - \sqrt{-1} J(x)$ and let $u(t) = u/|u|_{h(t)}$ for $t \ge 0$. Note that $|u|_{h(0)} = 1$.

Assume the conditions and notation of Proposition 3.2 and Proposition 3.1. Then we can see that by the propositions, the eigenvalues of the Ricci curvature of $h(t)$ with respect to $h(t)$ at o are $\mu_1 > \cdots > \mu_l > 0$ such that the multiplicity of μ_i is $\dim E_i(t)$ for t large enough.

Let $E_i^h(t)$ be the eigenspace of the Ricci tensor of $h(t)$ corresponding to the eigenvalue μ_i.

We want to prove the following:

Lemma 3.2. *With the assumptions as in Proposition* 3.2 *and with the above notations. Suppose $w_k(t) = \sum_{i=1}^l w_{k,i}(t)$ where $w_{k,i}(t)$ is the orthogonal projection of $w_k(t)$ onto $E_i(t+t_k)$ with respect to $g_k(t) = g(t_k+t)$ and suppose $u(t) = \sum_{i=1}^l u_i(t)$ where $u_i(t)$ is the orthogonal projection of $u(t)$ onto $E_i^h(t)$ with respect to $h(t)$. Then for any $T > 0$, the following are true:*

(i) *$w_k(t)$ converges uniformly to $u(t)$ on $t \in [0,T]$ in the sense that the real parts and the imaginary parts of $w_k(t)$ converge uniformly to the real part and imaginary part of $u(t)$ respectively.*

(ii) *$Rc_t^k(w_k(t), \bar{w}_k(t))$ converges uniformly to $Rc_t^h(u(t), \bar{u}(t))$ on $t \in [0,T]$ where Rc_t^k is the Ricci tensor of $g_k(t)$ at p and Rc_t^h is the Ricci tensor of $h(t)$ at o.*

(iii) *By passing to a subsequence if necessary, for $1 \le i \le l$, $|w_{k,i}(t)|_{g_k(t)}$ converge uniformly to $|u_i(t)|_{h(t)}$ on $t \in [0,T]$.*

Proof.

(i): Since $g_k(t)$ converges uniformly to $h(t)$ on $[0,T]$ at o and since $w_k \to u$, $|w_k|_{g_k(t)}$ converge to $|u|_{h(t)}$ uniformly on $[0,T]$. From this it is easy to see that (i) is true.

(ii): Since $g_k(t)$ converges uniformly on $U \times [0,T]$ in C^∞ norm, by (i) it is easy to see that (ii) is true.

(iii): Let $v_k^{(1)}, \ldots, v_k^{(n)}$ be a unitary basis for $T_p^{(1,0)}(M)$ with respect to $g_k(0)$. Passing to a subsequence if necessary, we may assume that they converge to a unitary basis $u^{(1)}, \ldots, u^{(n)}$ of $T_o^{(1,0)}(N)$ with respect to $h(0)$. Using the Gram-Schmidt process, we claim that we can obtain $v_k^{(1)}(t), \ldots, v_k^{(n)}(t)$ to be a unitary basis for $T_p^{(1,0)}(M)$ with respect to $g_k(t)$ and a unitary basis $u^{(1)}(t), \ldots, u^{(n)}(t)$ of $T_o^{(1,0)}(N)$ with respect to $h(t)$ such that $v_k^{(i)}(t)$ converges to $u^{(i)}(t)$ uniformly on $[0,T]$. Observe that since $g_k(t)$ converge to $h(t)$ uniformly on $[0,T]$ and $v_k^{(1)} \to u^{(1)}$, $|v_k^{(1)}|_{g_k(t)} \to |u^{(1)}|_{h(t)}$ uniformly on $[0,T]$. Thus if we define $v_k^{(1)}(t) = v_k^{(1)}/|v_k^{(1)}|_{g_k(t)}$ and $u^{(1)}(t) = u^{(1)}/|u^{(1)}|_{h(t)}$, then $v_k^{(1)}(t)$ converge to $u^{(1)}(t)$ uniformly on $[0,T]$. Now suppose we have found $v_k^{(i)}(t)$, $1 \le i \le m$ and $u^{(i)}(t)$, $1 \le i \le m$ such that (a) $v_k^{(i)}(t)$, $1 \le i \le m$ are unitary with respect to $g_k(t)$ and are linear combinations of $v_k^{(i)}$, $1 \le i \le m$;

(b) $u^{(i)}(t)$, $1 \leq i \leq m$ are unitary with respect to $h(t)$ and are linear combinations of $u^{(i)}$, $1 \leq i \leq m$; and (c) $v_k^{(i)}(t)$ converge to $u^{(i)}(t)$ uniformly on $[0, T]$ for $1 \leq i \leq m$. Define

$$v_k^{(m+1)}(t) = \frac{v_k^{(m+1)} - \sum_{i=1}^m \langle v_k^{(m+1)}, v_k^{(i)}(t)\rangle_{g_k(t)} v_k^{(i)}(t)}{|v_k^{(m+1)} - \sum_{i=1}^m \langle v_k^{(m+1)}, v_k^{(i)}(t)\rangle_{g_k(t)} v_k^{(i)}(t)|_{g_k(t)}}$$

and define

$$u^{(m+1)}(t) = \frac{u^{(m+1)} - \sum_{i=1}^m \langle u^{(m+1)}, u^{(i)}(t)\rangle_{h(t)} u^{(i)}(t)}{|u^{(m+1)} - \sum_{i=1}^m \langle u^{(m+1)}, u^{(i)}(t)\rangle_{h(t)} u^{(i)}(t)|_{h(t)}}$$

Then it is easy (a), (b) and (c) are still true with m replaced by $m+1$. Hence by induction, we can construct $v_k^{(i)}(t)$ and $u^{(i)}(t)$ as claimed.

Let $R^k_{i\bar{j}}(t) = Rc^k_t(v_k^{(i)}(t), \bar{v}_k^{(j)}(t))$ and let $R^h_{i\bar{j}}(t) = Rc^h_t(u^{(i)}(t), \bar{u}^{(j)}(t))$. Then as in (ii), we can prove that $R^k_{i\bar{j}}(t)$ converge to $R^h_{i\bar{j}}(t)$ uniformly on $[0, T]$. Denote by $P^k_i(t)$ the matrix with respect to the basis $v_k^{(1)}(t), \ldots, v_k^{(n)}(t)$ of the orthogonal projection onto $E_i(t+t_k)$ with respect to $g_k(t)$. Denote by $P_i(t)$ the matrix with respect to the basis $u^{(1)}(t), \ldots, u^{(n)}(t)$ of the orthogonal projection onto $E^h_i(t)$ with respect to $h(t)$. As in the proof of Proposition 3.1(iv),

$$P^k_s(t) = -\frac{1}{2\pi\sqrt{-1}} \int_C (R^k_{i\bar{j}}(t) - z\delta_{ij})^{-1} dz \tag{3.10}$$

and

$$P_s(t) = -\frac{1}{2\pi\sqrt{-1}} \int_C (R^h_{i\bar{j}}(t) - z\delta_{ij})^{-1} dz \tag{3.11}$$

where C is a circle on the complex plane with center at μ_s and radius ρ. Since $R^k_{i\bar{j}}(t)$ converge to $R^h_{i\bar{j}}(t)$ uniformly on $[0, T]$, (iii) follows from (3.10), (3.11) and (i). q.e.d.

4. Asymptotic behavior of Kähler Ricci flow (II)

Let $(M^n, \widetilde{g})$ be as in Theorem 2.1 with either maximal volume growth or positive curvature operator and let $g(x, t)$ be the corresponding solution to (2.3). As before, we denote the eigenvalues of $Rc(p, t)$ by $\lambda_i(t)$ for $i = 1, ..., n$ and we let μ_k, $E_k(t)$ and $P_k(t)$ for $k = 1, ..., l$ be as in Proposition 3.1. We let n_m for $m = 0, ..., l-1$ be such that $\lambda_k(t) \in (\mu_{m+1} - \rho, \mu_{m+1} + \rho)$ for all $n_m < k \leq n_{m+1}$ and t sufficiently large such that the intervals $[\mu_m - \rho, \mu_m + \rho]$ are disjoint as in Proposition 3.1 part (iv). For any nonzero vector $v \in T^{1,0}_p(M)$, let $v(t) = v/|v|_t$ where $|v|_t$ is the length of v with respect to $g(t)$ and $v_i(t) = P_i(t)v(t)$.

The goal of this section will be to prove that $Rc(p, t)$ can be 'diagonalized' simultaneously near infinity in a certain sense and that $g(t)$ is 'Lyapunov regular', to borrow a notion from dynamical systems (see

[**1**]). In the following lemmas we assume that the initial metric $\widetilde{g}(0)$ in (2.1), and thus by Proposition 3.1 $g(x,t)$ for all (x,t), has positive Ricci curvature.

Let $(N, h(t))$ be a gradient Kähler-Ricci soliton as in Proposition 3.2 and let $o \in N$, ϕ_t and $E_i^h(t)$ also be as in the Proposition. For any nonzero vector $w \in T_o^{1,0}(N)$ let $w(t) = w/|w|_{h(t)}$ and $w_i(t)$ be the projection of $w(t)$ onto $E_i^h(t)$. We begin by making the following observation.

Let ϕ_t be the flow along $-\frac{1}{2}\nabla f$ where $f_{i\bar{j}}(x) = R^h_{i\bar{j}}(x,0) + h_{i\bar{j}}(x,0)$ and $f_{ij} = 0$. Near o, we may choose local coordinates z_i such that $\partial_i = \frac{\partial}{\partial z_i}$ are unitary at o which diagonalize $f_{i\bar{j}}$ at o. We also assume that the origin corresponds to o. Then $\mu_1 > \mu_2 > \cdots > \mu_l > 0$ are distinct eigenvalues of Ric^h at $t = 0$ with respect to $h(0)$. Since ∂_i are eigenvectors of $f_{i\bar{j}}$, for each i we have

$$(\phi_t)_*(\partial_i) = e^{-\frac{1}{2}(\mu_j+1)t}\partial_i \tag{4.1}$$

for some j at o. Because of (4.1) and the fact that ∂_i are also eigenvectors of $R_{i\bar{j}}$ at o and $t = 0$, $E_i^h(0) = E_i^h(t)$ and $w_i(t) = w_i(0)/|w|_{h(t)}$.

Lemma 4.1. *Let $(N, h(t))$ be a gradient Kähler-Ricci soliton and $w \in T_o^{(1,0)}(N)$ with $|w|_{h(0)} = 1$ as above. Let $1 \le m < l$, and suppose $a < \sum_{j=m+1}^l |w_j(0)|^2_{h(0)} < 1 - a$ for some $0 < a < 1$. Then for $t \ge 0$,*

$$\frac{\sum_{j=m+1}^l |w_j(t)|^2_{h(t)}}{\sum_{j=1}^m |w_j(t)|^2_{h(t)}} \ge \frac{\sum_{j=m+1}^l |w_j(0)|^2_{h(0)}}{\sum_{j=1}^m |w_j(0)|^2_{h(0)}} \cdot e^{(\mu_m-\mu_{m+1})t}.$$

In particular,

$$\sum_{j=m+1}^l |w_j(t)|^2_{h(t)} \ge \sum_{j=m+1}^l |w_j(0)|^2_{h(0)}$$

for $t \ge 0$. Moreover, for any $\delta > 0$, there is a t_0 depending only on the a, μ_m, μ_{m+1} and δ such that for all $t \ge t_0$,

$$\sum_{j=m+1}^l |w_j(t)|^2_{h(t)} \ge 1 - \delta.$$

Proof. For simplicity, let us denote $|\cdot|_{h(t)}$ simply by $|\cdot|_t$.

$$\begin{aligned} |w_j(t)|_t^2 &= \frac{|(\phi_t)_*(w_j(0))|_0^2}{|(\phi_t)_*(w)|_0^2} \\ &= \frac{e^{(-\mu_j-1)t}|w_j(0)|_0^2}{|w|_t^2}. \end{aligned} \tag{4.2}$$

Hence for $t \geq 0$

(4.3)
$$\sum_{j=1}^{m} |w_j(t)|_t^2 = \frac{\sum_{j=1}^{m} e^{(-\mu_j-1)t}|w_j(0)|_0^2}{|w|_t^2} \leq \frac{e^{(-\mu_m-1)t}\sum_{j=1}^{m}|w_j(0)|_0^2}{|w|_t^2}$$

because $\mu_1 > \cdots > \mu_l$. Similarly,

$$\sum_{j=m+1}^{l} |w_j(t)|_t^2 = \frac{\sum_{j=m+1}^{l} e^{(-\mu_j-1)t}|w_j(0)|_0^2}{|w|_t^2} \geq \frac{e^{(-\mu_{m+1}-1)t}\sum_{j=m+1}^{l}|w_j(0)|_0^2}{|w|_t^2}. \tag{4.4}$$

The lemma then follows from (4.3) and (4.4). q.e.d.

Because of Proposition 3.2 and Lemma 3.2, we expect to have similar behavior for $g(t)$ for t large. More precisely, we have the following:

Lemma 4.2. *Let $v_k \in T_p^{(1,0)}(M)$ be a sequence such that $|v_k|_0 = 1$ for each k. Let $t_k \to \infty$ be a sequence in time. Define $f_{ik}(t) := |P_i(t)v_k(t)|_t^2$.*

(i) *Suppose there exists $a > 0$ and $1 \leq m \leq l$ for which*

$$\sum_{i \geq m} f_{ik}(t_k) \geq a \tag{4.5}$$

for all k. Then for any sequence $s_k > t_k$ we have

$$\liminf_{k\to\infty} \sum_{i\geq m} f_{ik}(s_k) \geq a. \tag{4.6}$$

(ii) *Suppose there exists $1 > a > 0$ and $1 \leq m \leq l$ for which*

$$a \leq \sum_{i \geq m} f_{ik}(t_k) \leq 1 - a. \tag{4.7}$$

for all k. Then for any $1 > \delta > 0$ there exists $T > 0$ such that

$$\liminf_{k\to\infty} \sum_{i\geq m} f_{ik}(t_k + T) \geq 1 - \delta. \tag{4.8}$$

Proof. Suppose (i) is false. Then $m > 1$ and there exists a subsequence of t_k which we will also denote by t_k, a sequence $s_k > t_k$, and some $\epsilon > 0$ for which

$$\sum_{i \geq m} f_{ik}(s_k) \leq a - \epsilon \tag{4.9}$$

for all k. Thus by the continuity of $f_{ik}(t)$ in t for each i (see Proposition 3.1(iv)), there is a sequence $t_k < T_k < s_k$ such that

$$\sum_{i \geq m} f_{ik}(T_k) = a - \frac{\epsilon}{2} \tag{4.10}$$

and

$$\sum_{i\geq m} f_{ik}(t) \leq a - \frac{\epsilon}{2} \tag{4.11}$$

for all $t \in [T_k, s_k]$.

Now define $g_k(t) = g(T_k + t)$. Then we may assume that $(M, g_k(t))$ converges to a soliton $(N, h(t))$ as in Proposition 3.2 such that p corresponds to the stationary point o. We may also assume that $v_k(T_k)$ converges to a vector w in $T_o^{1,0}(N)$ where w has length 1 in with respect to $h(0)$. Then by Lemma 3.2(iii), for any $T > 0$, we have

$$\lim_{k\to\infty} \sum_{i\geq m} f_{ik}(T_k + t) = \sum_{i\geq m} |w_i(t)|_{h(t)} \tag{4.12}$$

uniformly for all $t \in [0, T]$, where $w(t) = w/|w|_{h(t)}$ and $w_i(t)$ is the orthogonal projection of $w(t)$ onto the eigenspace of $Ric^h(t)$ at o of the eigenvalue μ_i with respect to $h(t)$.

We claim that $s_k - T_k > \tau$ for some $\tau > 0$. Otherwise, we may assume that $s_k - T_k \to 0$, and thus from (4.9), (4.10) and (4.12) we may draw the contradiction that

$$a - \frac{\epsilon}{2} = \sum_{i\geq m} |w_i(0)|_{h(0)} \leq a - \epsilon.$$

This proves the claim. Thus from (4.10), (4.11) and (4.12) we may conclude that

$$\sum_{i\geq m} w_i(0) = a - \frac{\epsilon}{2} \tag{4.13}$$

and

$$\sum_{i\geq m} w_i(t) \leq a - \frac{\epsilon}{2} \tag{4.14}$$

for all $t \in [0, \tau]$. But (4.13) and (4.14) contradict Lemma 4.1. This completes the proof of (i) by contradiction.

We now suppose (ii) is false. Note that $m > 1$ because $0 < a < 1$. Then there exists a $\delta > 0$ with the property that: given any $T > 0$, there exists a subsequence of t_k, which we also denote by t_k, for which

$$\sum_{i\geq m} f_{ik}(t_k + T) \leq 1 - \delta. \tag{4.15}$$

for all k.

Now we define $g_k(t) = g(t_k + t)$ and assume $(M, g_k(t))$ converges to a soliton $(N, h(t))$ as in the proof of (i). We also assume that $v_k(t_k)$ converges to a vector w in $T_o^{1,0}(N)$ where w has length 1 with respect to $h(0)$. Then by taking a limit as in the proof of (i), using Lemma 3.2(iii), (4.7) and (4.15), we have

$$a \leq \sum_{i \geq m} w_i(0) \leq 1 - a \tag{4.16}$$

and

$$\sum_{i \geq m} w_i(T) \leq 1 - \delta. \tag{4.17}$$

But for T sufficiently large depending only on a, μ_{m-1}, μ_m and δ, (4.16) and (4.17) contradict Lemma 4.1. This complete our proof of (ii) by contradiction. q.e.d.

We are ready to prove the main theorem in this section.

Theorem 4.1. *Let $(M^n, \widetilde{g})$ be as in Theorem 2.1 with either maximal volume growth or positive curvature operator, and let $g(x,t)$ be the corresponding solution to (2.3). With the same notation as in the beginning of this section, $V = T_p^{(1,0)}(M)$ can be decomposed orthogonally with respect to $g(0)$ as $V_1 \oplus \cdots \oplus V_l$ so that the following are true:*

(i) *If v is a nonzero vector in V_i for some $1 \leq i \leq l$, then $\lim_{t\to\infty} |v_i(t)|_t = 1$ and thus $\lim_{t\to\infty} Rc(v(t), \bar{v}(t)) = \mu_i$ and*

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{|v|_t^2}{|v|_0^2} = -\mu_i - 1.$$

Moreover, the convergences are uniform over all $v \in V_i \setminus \{0\}$.

(ii) *For $1 \leq i, j \leq l$ and for nonzero vectors $v \in V_i$ and $w \in V_j$ where $i \neq j$, $\lim_{t\to\infty} \langle v(t), w(t)\rangle_t = 0$ and the convergence is uniform over all such nonzero vectors v, w.*

(iii) $\dim_{\mathbb{C}}(V_i) = n_i - n_{i-1}$ *for each i.*

(iv)
$$\sum_{i=1}^{l} (-\mu_i - 1) \dim_{\mathbb{C}} V_i = \lim_{t\to\infty} \frac{1}{t} \log \frac{\det(g_{i\bar{j}}(t))}{\det(g_{i\bar{j}}(0))}.$$

Proof. We first assume that the initial metric $\widetilde{g}(0)$ in (2.1), and thus $g(x,t)$ for all (x,t), has positive Ricci curvature by Proposition 3.1.

To prove (i), let $v \in T_p(M)$ be a fixed nonzero vector and let $f_i(t) = |v_i(t)|_t^2$. We claim that $\lim_{t\to\infty} f_m(t) = 1$ for some m, and thus

$$\lim_{t\to\infty} f_k(t) = 0$$

for all $k \neq m$. To prove our claim it will be sufficient to prove the following for every m (by (ii) of the previous Lemma): Suppose $\lim_{t\to\infty} f_j(t) = 0$ for all $j < m$. Then either

$$\lim_{t\to\infty} f_m(t) = 1 \tag{4.18}$$

or

$$\lim_{t\to\infty} f_m(t) = 0. \tag{4.19}$$

If $m = l$, then we must have $\lim_{t\to\infty} f_m(t) = 1$ under the supposition. Suppose $1 \le m < l$ and $\lim_{t\to\infty} f_j(t) = 0$ for all $j < m$ and that neither (4.18) nor (4.19) holds. By the continuity of $f_i(t)$, we can find $t_k \to \infty$ such that

$$a \le \sum_{i\ge m+1} f_i(t_k) \le 1 - a \tag{4.20}$$

for some $0 < a < 1$. By letting $v_k = v$ for all k, it follows from Lemma 4.2(ii), we can find $T > 0$, such that passing to a subsequence if necessary we have

$$\sum_{i\ge m+1} f_i(t_k + T) \ge 1 - \frac{a}{2}. \tag{4.21}$$

For each j, we can find k_j such that $t_{k_j} > t_j + T$. Since

$$\sum_{i\ge m+1} f_i(t_j + T) \ge 1 - \frac{a}{2}$$

and

$$\sum_{i\ge m+1} f_i(t_{k_j}) \le 1 - a$$

for all j, we may derive a contradiction from part (i) of Lemma 4.2. Thus our initial assumption was false, and for any $v \in T_p(M)$ and m, either (4.18) or (4.19) holds. Thus for any nonzero $v \in T_p(M)$ we have $\lim_{t\to\infty} f_m(t) = 1$ for some m

Now suppose $\lim_{t\to\infty} f_m(t) = 1$. Using (2.3), Proposition 3.1, the definition of μ_i and the definition of $f_i(t)$, a straight forward calculation gives

$$\lim_{t\to\infty} \frac{1}{t} \log |v|_t^2 = -\mu_m - 1.$$

Note that if

$$\lim_{t\to\infty} \frac{1}{t} \log |v|_t^2 = -\mu_i - 1$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log |w|_t^2 = -\mu_j - 1$$

and $i \le j$ (so that $-\mu_j \ge -\mu_i$), then

$$\lim_{t\to\infty} \frac{1}{t} \log |av + bw|_t^2 \le -\mu_j - 1. \tag{4.22}$$

provided $av + bw \ne 0$.

Let V_1 be the subspace of $V = T_p^{(1,0)}(M)$ defined by

$$V_1 = \left\{ v \in V \setminus \{0\} \middle|\ \lim_{t\to\infty} \frac{1}{t} \log |v|_t^2 = -\mu_1 - 1 \right\} \cup \{0\}.$$

It is easy to see that V_1 is a subspace by (4.22). Let $V_1^\perp$ be the orthogonal complement of V_1 with respect to $g(0)$. Then by the definition of V_1, for any nonzero $v \in V_1^\perp$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log |v|_t^2 = -\mu_j - 1$$

for some $j > 1$. Define

$$V_2 = \left\{ v \in V_1^\perp \setminus \{0\} | \lim_{t\to\infty} \frac{1}{t} \log |v|_t^2 = -\mu_2 - 1 \right\} \cup \{0\}.$$

Continuing in this way, we can decompose V as $V = V_1 \oplus \cdots \oplus V_l$ orthogonally with respect to $g(0)$, such that if $v \in V_m$, then

$$\lim_{t\to\infty} f_m(t) = 1, \tag{4.23}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{|v|_t^2}{|v|_0^2} = -\mu_m - 1. \tag{4.24}$$

It remains to prove that both convergences are uniform on $V_m \setminus \{0\}$. It is sufficient to prove the convergence in (4.23) is uniform. Suppose the convergence is not uniform over $V_m \setminus \{0\}$. Then there exist $v_k \in V_m$, $t_k \to \infty$, $\epsilon > 0$ such that $|v_k|_0 = 1$, v_k converge to some vector $v \in V_m$ and

$$f_{mk}(t_k) = |P_m(t_k)v_k(t_k)|_{t_k}^2 \le 1 - 5\epsilon. \tag{4.25}$$

Since $f_{mk}(t) = |P_m(t)v_k(t)|_t^2 \to 1$ as $t \to \infty$ for all k, we can find $r_k > t_k$ such that

$$f_{mk}(r_k) \ge 1 - \epsilon. \tag{4.26}$$

On the other hand, for each fixed s, $\lim_{k\to\infty} f_{mk}(s) = |P_m(s)v(s)|_s^2$. Moreover, $\lim_{s\to\infty} |P_m(s)v(s)|_s^2 = 1$ because $v \in V_m$ and $|v|_0 = 1$. Hence passing to a subsequence if necessary, we can find $s_k \to \infty$ such that $s_k < t_k$ and

$$f_{mk}(s_k) \ge 1 - \epsilon. \tag{4.27}$$

Now we claim that there exists k_0 such that if $k \ge k_0$ then

$$\sum_{i\ge m} f_{ik}(t) \ge 1 - 2\epsilon \tag{4.28}$$

for all $t > s_k$. Otherwise, we can find $s_k' > s_k$ for infinitely many k such that

$$\sum_{i\ge m} f_{ik}(s_k') \le 1 - 2\epsilon. \tag{4.29}$$

But (4.27), (4.29) and the fact that $s_k' > s_k$ contradicts Lemma 4.2(i). Hence (4.28) is true.

If $m = l$, then for $k \geq k_0$, (4.28) contradicts (4.25) because $t_k > s_k$. Suppose $1 \leq m < l$, then by (4.25) and (4.28), for $k \geq k_0$, we have

$$\sum_{i \geq m+1} f_{mk}(t_k) \geq 3\epsilon. \tag{4.30}$$

and from (4.26)

$$\sum_{i \geq m+1} f_{mk}(r_k) \leq \epsilon. \tag{4.31}$$

for k large enough. Since $r_k > t_k$, (4.30) and (4.31) contradicts Lemma 4.2(i) again. This completes the proof of part (i).

Part (ii) of the theorem follows directly from the definition of $v(t)$ and $w(t)$, the orthogonality of the spaces $E_i(t)$ with respect to $g(t)$ and part (i).

To prove (iii), we begin by showing the following: Fix $1 \leq m \leq l$. Let $v_k \in E_1(s_k) + \cdots + E_m(s_k)$ with $s_k \to \infty$ such that $|v_k|_0 = 1$ and v_k converge to a vector $u \in T_p^{(1,0)}(M)$ of unit length with respect to $g(0)$. Then

$$\lim_{t \to \infty} |u_j(t)|_t = 0 \tag{4.32}$$

for all $j > m$, where $u_j(t) = P_j(t)u(t)$ and $u(t) = u/|u|_t$ as before.

Suppose this is false. Then by (i), we have

$$\lim_{t \to \infty} \sum_{j \geq m+1} |u_j(t)|_t^2 = 1. \tag{4.33}$$

Let $f_{jk}(t) = |P_j(t)v_k(t)|_t^2$. Since for fixed t,

$$\lim_{k \to \infty} f_{jk}(t) = |u_j(t)|_t^2,$$

as before, given any $\frac{1}{2} > \epsilon > 0$ we may choose a subsequence of s_k also denoted by s_k, and a sequence $t_k < s_k$ for which $t_k \to \infty$ and

$$\sum_{j \geq m+1} f_{jk}(t_k) \geq 1 - \epsilon \tag{4.34}$$

for all k. But $\sum_{j \geq m+1} f_{jk}(s_k) = 0$ for all k by definition. This is impossible by Lemma 4.2(i). Thus (4.32) is true for all $j > m$.

We now show that for all $1 \leq m \leq l$, $\dim_{\mathbb{C}} V_m = n_m - n_{m-1}$ which is equal to $\dim_{\mathbb{C}} E_m(t)$ for t large enough. Let $d_i = \dim V_i$. We claim that for any $1 \leq m \leq l$,

$$d_1 + \cdots + d_m \geq n_m. \tag{4.35}$$

Fix $1 \leq m \leq l$. Choose $t_k \to \infty$. We may assume that $\dim E_j(t_k) = n_j - n_{j-1}$ for all j and k. Hence we can choose a basis $v_1(t_k), \ldots, v_{n_m}(t_k)$ of $\sum_{j=1}^{m} E_j(t_k)$. Using Gram-Schmidt process, we may assume that

$$v_1(t_k)/|v_1(t_k)|_{g(0)}, \ldots, v_{n_m}(t_k)/|v_{n_m}(t_k)|_{g(0)}$$

are unitary with respect to $g(0)$. Moreover, we may assume that for $k \to \infty$, $v_j(t_k)/|v_j(t_k)|_0$ converge to some w_j for all $1 \le j \le n_m$. Hence we have n_m vectors $w_1, \dots, w_{n_m}$. They satisfy the following:

(a) They are unitary with respect to $g(0)$ by construction.

(b) For each $1 \le j \le n_m$

$$\lim_{t\to\infty} \frac{1}{t} \log |w_j(t)|_t^2 \le -\mu_m - 1$$

by (4.32).

For each w_j $(1 \le j \le n_m)$, $w_j = \sum_{k=1}^{l} w_{j,k}$ where $w_{j,k} \in V_k$. If there is a $k > m$ such that $w_{j,k} \neq 0$, then by (i) and the fact that $-\mu_k > -u_m$ and the definition of V_k, we have

$$\lim_{t\to\infty} \frac{1}{t} \log |w_j(t)|_t^2 \ge -\mu_k - 1 > -\mu_m - 1,$$

contradicting (b). Thus $w_j \in V_1 \oplus \cdots \oplus V_m$ for $1 \le j \le n_m$. From this (4.35) follows because the w_j are linearly independent by (a).

Choose a unitary basis $v_{j,1}, \dots, v_{j,d_j}$ of V_j with respect to $g(0)$ for all $1 \le j \le l$. This gives a unitary basis with respect to $g(0)$ for $T_p^{(1,0)}(M)$. Let $g_{i\bar{j}}(t)$ be components of $g(t)$ with respect to this basis. Then

$$\det(g_{i\bar{j}}(t)) \le \prod_{j=1}^{l} \prod_{k=1}^{d_j} |v_{j,k}|_{g(t)}^2.$$

Since $\lim_{t\to\infty} R(t) = \sum_{j=1}^{l} (n_j - n_{j-1})\mu_j$ by Proposition 3.1 where $R(t)$ is the scalar curvature of $g(t)$, by (2.3) and the above inequality we have

$$\begin{aligned} \sum_{j=1}^{l} (n_j - n_{j-1})(-\mu_j - 1) &= -\lim_{t\to\infty} R(t) - n \\ &= \lim_{t\to\infty} \frac{1}{t} \log \frac{\det(g_{i\bar{j}}(t))}{\det(g_{i\bar{j}}(0))} \\ &\le \sum_{j=1}^{l} \sum_{k=1}^{d_j} \lim_{t\to\infty} \frac{1}{t} \log |v_{j,k}|_{g(t)}^2 \\ &= \sum_{j=1}^{l} d_j(-\mu_j - 1). \end{aligned} \tag{4.36}$$

Let us denote $n_j - n_{j-1}$ by k_j, then we have

$$\sum_{j=1}^{l} k_j(-\mu_j) \le \sum_{j=1}^{l} d_j(-\mu_j)$$

and $\sum_{j=1}^{m} d_j \ge \sum_{j=1}^{m} k_j$ for all $1 \le m \le l$ by (4.35). Also $\sum_{j=1}^{l} d_j = \sum_{j}^{l} k_j = n$. Since $-\mu_1 < -\mu_2 < \cdots < -\mu_l$, we must have $d_j = k_j$ for all

j. In fact, if this is not the case, since $d_1 \geq k_1$, and $\sum_{j=1}^m d_j \geq \sum_{j=1}^m k_j$ for all $1 \leq m \leq l$, then we can find m to be the first m such that $d_m > k_m$ and $d_j = k_j$ for $j < m$. We have

$$
\begin{aligned}
(4.37)\quad & \sum_{j=1}^{l} d_j(-\mu_j) \\
&= \sum_{j<m} k_j(-\mu_j) + k_m(-\mu_m) + (d_m - k_m)(-\mu_m) + \sum_{j>m} d_j(-\mu_j) \\
&< \sum_{j\leq m} k_j(-\mu_j) + (d_m - k_m + d_{m+1})(-\mu_{m+1}) + \sum_{j>m+1} d_j(-\mu_j)
\end{aligned}
$$

because $-\mu_m < -\mu_{m+1}$ and $d_m - k_m > 0$. If we let $d_j' = k_j$ for $1 \leq j \leq m$, $d_j' = d_j$ for $j > m+1$, and $d_{m+1}' = d_m - k_m + d_{m+1}$ then we have

$$
\sum_{j=1}^{l} k_j(-\mu_j) < \sum_{j=1}^{l} d_j'(-\mu_j)
$$

and $\sum_{j=1}^p d_j' \geq \sum_{j=1}^p k_j$ for all $1 \leq p \leq l$ by (4.35). Also $\sum_{j=1}^l d_j' = \sum_j^l k_j = n$, and $d_j' = k_j$ for all $1 \leq j \leq m$. By induction, we will end up with

$$
\sum_{j=1}^{l} k_j(-\mu_j) < \sum_{j=1}^{l} k_j(-\mu_j)
$$

which is impossible. This completes the proof of part (iii).

Part (iv) follows directly from part (iii) and the first two equalities in (4.36).

We have thus proved that Theorem in the case that (M, g) satisfied the additional assumption of positive Ricci curvature. Now if the Ricci curvature is not strictly positive on M, we can use the results in [**5**] to reduce back to the case of positive Ricci curvature. This completes the proof of the theorem. q.e.d.

5. Uniformization

Let $(M^n, \widetilde{g})$ be as in Theorem 2.1 and assume $(M, \widetilde{g})$ has either maximum volume growth or positive curvature operator. Let $\widetilde{g}(t)$ be the solution of the Kähler-Ricci flow (2.1) and let $g(t)$ be the corresponding solution of the normalized flow (2.3). Fix a point $p \in M$. Then by Corollary 2.2, there exist $1 > r_1$ and $r_2 > 0$ such that for all $t > 0$, there is a holomorphic map $\Phi_t : D(r_1) \to M$ (where $D(r_1) = \{z \in \mathbb{C}^n | \ |z| < r_1\}$), satisfying the following:

(5.1)

$$\begin{cases} \Phi_t \text{ is biholomorphism from } D(r_1) \text{ onto its image.} \\ \Phi_t(0) = p. \\ \Phi_t^*(g(t))(0) = g_\epsilon, \text{ where } g_\epsilon \text{ is the standard Euclidean metric of } \mathbb{C}^n. \\ \frac{1}{r_2} g_\epsilon \le \Phi_t^*(g(t)) \le r_2 g_\epsilon \text{ in } D(r_1). \end{cases}$$

By Corollary 2.2, we will choose $T > 0$ such that if $F_{i+1} = \Phi_{(i+1)T}^{-1} \circ \Phi_{iT}$, then for each i, F_i is a holomorphic map from $D(r_1)$ into $\mathbb{C}^n$ and is a biholomorphism onto its image. Moreover,

$$F_i(D(r_1)) \subset D(r_1), \ |F_i(z)| \le \delta |z| \text{ for some } 0 < \delta < 1. \tag{5.2}$$

Let $A_i = F_i'(0)$ be the Jacobian matrix of F_i at 0. Since $R_{i\bar{j}} \ge 0$ for all t and is uniformly bounded, we have

$$a|v| \le |A_i(v)| \le b|v| \tag{5.3}$$

for some $0 < a < b < 1$ for all i. Here a, b, δ are independent of i. We will now modify(decompose) the maps F_i as in [**36**] and [**23**], then assemble them to obtain a global biholomorphism from M to $\mathbb{C}^n$.

We begin by fixing some notation. As in Proposition 3.1, let $0 \le \lambda_1(t) \le \lambda_2(t) \le \ldots \le \lambda_n(t)$ be the eigenvalues of $R_{i\bar{j}}(t)$ with respect to $g(t)$ and let $0 \le \mu_1 < \mu_2 \cdots < \mu_l$ be their limits. Let $\rho > 0$ and $E_k(t)$, $1 \le k \le l$ also be as in Proposition 3.1 and let $P_k(t)$ be the orthogonal projection onto $E_k(t)$ with respect to $g(t)$. Let $\tau_k = e^{-(\mu_k+1)T}$, $1 \le k \le l$. Note that for convenience, we have reversed the order of λ_i and hence the order of μ_k.

By Theorem 4.1, $T_p^{(1,0)}(M)$ can be decomposed orthogonally with respect to the initial metric as $E_1 \oplus \cdots \oplus E_l$ such that if $v \in E_k$ and $w \in E_j$ are nonzero vectors and if $v(t) = v/|v|_t$, $w(t) = w/|w|_t$ where $|\cdot|_t$ is the norm taken with respect to $g(t)$, then for $1 \le k \le l$ and for $j \ne k$

$$\lim_{t\to\infty} |P_k(t)v(t)|_t = 1, \ \text{ and } \ \lim_{t\to\infty} \langle v(t), w(t)\rangle_t = 0. \tag{5.4}$$

where $\langle \cdot, \cdot \rangle_t$ is the inner produce with respect to $g(t)$. Moreover, the convergences are uniform over all nonzero vectors in E_j and E_k.

For any i, let $E_{i,k} = d\Phi_{iT}^{-1}(E_k)$, $1 \le k \le l$. Denote $A(i) = A_i \cdots A_1$ and $A(i+j, i) = A_{i+j} \cdots A_{i+1}$. Then $E_{i,k} = A(i)(E_{1,k})$ and $A_{i+1}(E_{i,k}) = E_{i+1,k}$.

Lemma 5.1. *Given $\epsilon > 0$, there exists i_0 such that if $i \ge i_0$, then the following are true:*

(i) $(1-\epsilon)\tau_k |v|^2 \le |A_i(v)|^2 \le (1+\epsilon)\tau_k |v|^2$ *for all $v \in E_{i,k}$ and $1 \le k \le l$, where $\tau_k = e^{-(\mu_k+1)T}$.*

(ii) *For any nonzero vector* $v \in \mathbb{C}^n$

$$(1-\epsilon) \le \frac{|v|^2}{\sum_{k=1}^{l} |v_k|^2} \le (1+\epsilon)$$

where $v = \sum_{k=1}^{l} v_k$ *is the decomposition of* v *in* $E_{i,1} \oplus \cdots \oplus E_{i,l}$.

Proof.

(i) Let $1 \le k \le l$. By (5.4), given $\epsilon > 0$, there exists $t_0 > 0$ such that

$$|P_k(t)(w)|_t \ge 1 - \epsilon$$

for all $w \in E_k \setminus \{0\}$ and for all $t \ge t_0$. By the definition of E_k and Proposition 3.1, we have that $|\mathrm{Ric}(w(t), \bar{w}(t)) - \mu_k| \le \epsilon$ for all $w \in E_k \setminus \{0\}$, provided t_0 is large enough. Suppose $i_0 > t_0/T$. Then for $i \ge i_0$ and $v \in E_{i,k} \setminus \{0\}$, there is $w \in E_k \setminus \{0\}$ with $d\Phi_{iT}^{-1}(w) = v$. Hence $A_i(v) = d\Phi_{(i+1)T}^{-1}(w)$. By (5.1), $|v| = |w|_{iT}$ and $|A_i(v)| = |w|_{(i+1)T}$. By the Kähler-Ricci flow equation we have

$$\begin{aligned}\log\left[\frac{|A_i(v)|^2}{|v|^2}\right] + (\mu_k + 1)T &= \log\left[\frac{|w|^2_{(i+1)T}}{|w|^2_{iT}}\right] + (\mu_k + 1)T \\ &= \int_{iT}^{(i+1)T} (\mu_k - \mathrm{Ric}(w(t), \bar{w}(t)))\, dt.\end{aligned}$$

Since $|\mathrm{Ric}(w(t), \bar{w}(t)) - \mu_k| \le \epsilon$ and T is fixed, it is easy to see that (i) is true.

(ii) Let $v \in \mathbb{C}^n$ be nonzero and let $v = \sum_{k=1}^{l} v_k$ be the decomposition of v in $E_{i,1} \oplus \cdots \oplus E_{i,l}$. Let $w \in T_p^{(1,0)}(M)$ be such that $d\Phi_{iT}^{-1}(w) = v$ and similarly decompose $w = \sum_{k=1}^{l} w_k$ with respect to $E_1 \oplus \cdots \oplus E_l$. Then $v_k = d\Phi_{iT}^{-1}(w)$. Since $\langle v_j, v_k \rangle = \langle w_j, w_k \rangle_{g(iT)}$ and $|v|^2 = |w|^2_{g(iT)}$ by (5.1), (ii) follows from (5.4). q.e.d.

Let us fix more notation. Let Φ be a polynomial map from $\mathbb{C}^n$ into $\mathbb{C}^n$, which means that each component of Φ is a polynomial. Suppose Φ is of homogeneous of degree m. That is to say, each component of Φ is a homogeneous polynomial of degree $m \ge 1$. We define

$$\|\Phi\| = \sup_{v \in \mathbb{C}^n, v \ne 0} \frac{|\Phi(v)|}{|v|^m}.$$

In general, if Φ is a polynomial map with $\Phi(0) = 0$, let $\Phi = \sum_{m=1}^{q} \Phi_m$ be the decomposition of Φ such that Φ_m is homogeneous of degree m, then $\|\Phi\|$ is defined as

$$\|\Phi\| = \sum_{m=1}^{q} \|\Phi_m\|.$$

If we decompose $\mathbb{C}^n$ as $E_{i,1} \oplus \cdots \oplus E_{i,l}$, we will denote $\mathbb{C}^n$ by $\mathbb{C}^n_i$. Let $\Phi : \mathbb{C}^n_i \to \mathbb{C}^n_{i+1}$ be a map. Then we decompose Φ as $\Phi(v) =$

$\sum_{k=1}^{l} \Phi_k(v) = \Phi_1 \oplus \cdots \oplus \Phi_l$ where $\Phi_k(v) \in E_{i+1,k}$. Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a multi-index such that $|\alpha| = \sum_{k=1}^{l} \alpha_k = m \geq 1$. Then a polynomial map Φ is said to be *homogeneous of degree* α if

$$\Phi(c_1 v_1 \oplus \cdots \oplus c_l v_l) = c^\alpha \Phi(v_1 \oplus \cdots \oplus v_l),$$

where $v_k \in E_{i,k}$. Note that if Φ homogeneous of degree α, then Φ is homogeneous of degree $|\alpha|$ in the usual sense. Φ is said to be *lower triangular*, if $\Phi_k(v_1 \oplus \cdots \oplus v_l) = c_k v_k + \Psi_k(v_1 \oplus \cdots \oplus v_{k-1})$.

Lemma 5.2. *Let $\Phi : \mathbb{C}_i^n \to \mathbb{C}_{i+1}^n$ be homogeneous of degree $\alpha = (\alpha_1, \ldots, \alpha_l)$ with $|\alpha| = m$. Then*

$$|\Phi(v_1 \oplus \cdots \oplus v_l)| \leq l^m \|\Phi\| \, |v_1|^{\alpha_1} \cdots |v_l|^{\alpha_l}.$$

Here by convention if $\alpha_i = 0$, then $|v_i|^{\alpha_i} = 1$ for all v_i.

Proof. Let $v = v_1 \oplus \cdots \oplus v_l$ such that $|v_k| = 1$ for all $1 \leq k \leq l$, then

$$|\Phi(v)| \leq \|\Phi\| \, |v|^m \leq l^m \|\Phi\|.$$

Hence if $v_k \neq 0$ for all k, then

$$|\Phi(v)| = |\Phi(|v_1| \frac{v_1}{|v_1|} \oplus \cdots \oplus |v_l| \frac{v_l}{|v_l|})| \leq l^m \|\Phi\| \, |v_1|^{\alpha_1} \cdots |v_l|^{\alpha_l}.$$

From this the lemma follows. q.e.d.

Note that $\tau_1 > \cdots > \tau_l$. Choose $1 > \epsilon > 0$ small enough such that $b^2(1-\epsilon)^{-1}(1+\epsilon) < 1$ where $b < 1$ is the constant in (5.3). Since we are interested in the maps F_i for large i, without loss of generality, we assume the conclusions of Lemma 5.1 are true for all i with this ϵ. Let $m_0 \geq 2$ be a positive integer such that $a^{-1} b^{m_0} < \frac{1}{2}$, where $0 < a < b < 1$ are the constants in (5.3).

We now begin to assemble the maps F_i to produce a global biholomorphism from M to $\mathbb{C}^n$. The constructions follow those in [**36**] and [**23**]; in particular those in [**23**] where the authors study the dynamics of a randomly iterated sequence of biholomorphisms.

Lemma 5.3. *Let $\Phi_{i+1} : \mathbb{C}_i^n \to \mathbb{C}_{i+1}^n$, $1 \leq i < \infty$, be a family homogeneous polynomial maps of degree $m \geq 2$ such that* $\sup_i \|\Phi_i\| < \infty$. *Then there exist homogeneous polynomial maps H_{i+1} and Q_{i+1} of degree m from $\mathbb{C}_i^n$ to $\mathbb{C}_{i+1}^n$ such that $\Phi_{i+1} = Q_{i+1} + H_{i+1} - A_{i+2}^{-1} H_{i+2} A_{i+1}$. Moreover, H_{i+1} and Q_{i+1} satisfy the following:*

(i) $\sup_i \|H_i\| < \infty$ *and* $\sup_i \|Q_i\| < \infty$.
(ii) $Q_{i+1} = 0$ *if $m \geq m_0$.*
(iii) Q_{i+1} *is lower triangular:*

$$\begin{aligned} &Q_{i+1}(v_1 \oplus \cdots \oplus v_l) \\ &= 0 \oplus Q_{i+1,2}(v_1) \oplus Q_{i+1,3}(v_1 \oplus v_2) \oplus \cdots \oplus Q_{i+1,l}(v_1 \oplus \cdots \oplus v_{l-1}) \end{aligned}$$

where $v_k \in E_{i,k}$ and $Q_{i+1,k} : \mathbb{C}_i^n \to E_{i+1,k}$.

Proof. For each i, let β_k be a unitary basis for $E_{i,k}$ with respect to the standard metric of $\mathbb{C}_i^n$. Let $v \in \mathbb{C}_i^n$ and if $v = \sum_{k=1}^l \sum_{w\in\beta_k} a_w w$, then

$$C_1^{-1}|v|^2 \le \sum_{k=1}^{l}\sum_{w\in\beta_k} |a_w|^2 \le C_1|v|^2$$

for some constant C_1 independent of i by Lemma 5.1(ii). Hence if we decompose Φ_{i+1} into α-homogeneous parts $\Phi_{i+1,\alpha}$, $|\alpha| = m$, then $\|\Phi_{i+1,\alpha}\| \le C_2\|\Phi_{i+1}\|$ for some constant C_2 independent of Φ_{i+1} and i. Moreover, if we decompose $\Phi_{i+1,\alpha} = \Phi_{i+1,\alpha,1} \oplus \cdots \oplus \Phi_{i+1,\alpha,l}$ with $\Phi_{i+1,\alpha,k}(v) \in E_{i+1,k}$, then by Lemma 5.1(ii) again,

$$\|\Phi_{i+1,\alpha,k}\| \le C_3\|\Phi_{i+1,\alpha}\|$$

for some constant C_3 independent of i. Hence in order to prove the lemma, we may assume that Φ_{i+1} is homogeneous of degree

$$\alpha = (\alpha_1, \ldots, \alpha_l)$$

with $|\alpha| = m$ and $\Phi_{i+1}(v) \in E_{i+1,k}$ for all i for some $1 \le k \le l$.

Suppose $m \ge m_0$. Then we define $Q_{i+1} = 0$ and let

$$H_{i+1} = \Phi_{i+1} + \sum_{s=0}^{\infty} A_{i+2}^{-1}\cdots A_{i+s+2}^{-1}\Phi_{i+s+2}A_{i+s+1}\cdots A_{i+1}.$$

To see H_{i+1} is well-defined, by (5.3) we have that for any $v \in \mathbb{C}_i^n$,

$$|\Phi_{i+s+2}A_{i+s+1}\cdots A_{i+1}(v)| \le \|\Phi_{i+s+2}\|(b^{s+1}|v|)^m$$

and

$$\begin{aligned}|A_{i+2}^{-1}\cdots A_{i+s+2}^{-1}\Phi_{i+s+2}A_{i+s+1}\cdots A_{i+1}(v)| &\le \|\Phi_{i+s+2}\|\left(a^{-1}b^m\right)^{s+1}|v|^m\\ &\le 2^{-s-1}\|\Phi_{i+s+2}\|\,|v|^m.\end{aligned}$$

Hence H_{i+1} is well-defined, homogeneous of degree m and $\|H_{i+1}\| \le C_4$ for some constant C_4 independent of i. It is easy to see that H_{i+1} and Q_{i+1} satisfy the required conditions.

Now suppose $2 \le m < m_0$. Decompose Φ_{i+1} as $\Phi_{i+1}^{(1)} + \Phi_{i+1}^{(2)}$ where $\Phi_{i+1}^{(1)}(v_1 \oplus \cdots \oplus v_l) = \Phi_{i+1}(v_1 \oplus \cdots \oplus v_{k-1} \oplus 0 \cdots \oplus 0)$ consisting of terms that depending only on $v_1, \ldots, v_{k-1}$ and $\Phi_{i+1}^{(2)} = \Phi_{i+1} - \Phi_{i+1}^{(1)}$. Let $Q_{i+1} = \Phi_{i+1}^{(1)}$. Since $\Phi_{i+1}(v) \in E_{i+1,k}$, it is easy to see that Q_{i+1} satisfies condition (iii) in the lemma. It is also easy to see that $\|Q_{i+1}\| \le \|\Phi_{i+1}\|$.

Suppose $\alpha_j = 0$ for all $j \ge k$. Then $\Phi_i^{(2)} = 0$, and in this case we let $H_{i+1} = 0$. Then Q_{i+1} and H_{i+1} satisfy the required conditions.

Suppose there is $j \ge k$ with $\alpha_j \ge 1$. Then define

$$H_{i+1} = \Phi_{i+1}^{(2)} + \sum_{s=0}^{\infty} A_{i+2}^{-1}\cdots A_{i+s+2}^{-1}\Phi_{i+s+2}^{(2)}A_{i+s+1}\cdots A_{i+1}. \tag{5.5}$$

To prove H_{i+1} is well-defined and $\|H_{i+1}\|$ is uniformly bounded, we observe that

$$\|\Phi^{(2)}_{i+s+2}\| \le \|\Phi^{(1)}_{i+s+2}\| + \|\Phi_{i+s+2}\| \le 2\|\Phi_{i+s+2}\|. \tag{5.6}$$

Let $v \in \mathbb{C}^n_i$ and let $w = w_1 \oplus \cdots \oplus w_l = A(i+s+1, i)(v)$ and let $u = A(i+s+2, i+1)^{-1}(\Phi^{(2)}_{i+s+2}(w))$. Note that if $v = v_1 \oplus \cdots \oplus v_l$ with $v_q \in E_{i,q}$, then $A_{i+r}(v_q) \in E_{i+r,q}$. Hence by Lemma 5.2, Lemma 5.1(i) and (5.3)

$$\begin{aligned} |\Phi^{(2)}_{i+s+2}(w)| &\le l^m \|\Phi^{(2)}_{i+s+2}\|\, |w_1|^{\alpha_1} \ldots |w_l|^{\alpha_l} \\ &\le 2l^m \|\Phi_{i+s+2}\| |w|^{m-1}\, |w_j| \\ &\le 2l^m \|\Phi_{i+s+2}\| b^{(s+1)(m-1)} \,[(1+\epsilon)\tau_j]^{\frac{s+1}{2}}\ |v|^m. \end{aligned}$$

Since $\Phi^{(2)}_{i+s+2}\, w \in E_{i+s+2,\,k}$, by Lemma 5.1(i) and the fact that $A^{-1}_{r+1}(E_{r+1,k}) = E_{r,k}$ for all r, we have

$$\begin{aligned} |u| &= |A(i+s+2, i+1)^{-1}(\Phi^{(2)}_{i+s+2} w)| \\ &\le [(1-\epsilon)\tau_k]^{-\frac{s+1}{2}}\ |\Phi^{(2)}_{i+s+2} w| \\ &\le 2l^m \|\Phi_{i+s+2}\| b^{(s+1)(m-1)}\, [(1-\epsilon)\tau_k]^{-\frac{s+1}{2}}\ [(1+\epsilon)\tau_j]^{\frac{s+1}{2}}\ |v|^m \\ &\le 2l^m \|\Phi_{i+s+2}\| \left[b^2(1-\epsilon)^{-1}(1+\epsilon)\right]^{\frac{s+1}{2}} \end{aligned} \tag{5.7}$$

since $\tau_k \ge \tau_j$ for $j \ge k$, $m \ge 2$ and $b < 1$. Since we have chosen ϵ such that $b^2(1-\epsilon)^{-1}(1+\epsilon) < 1$, from (5.5)–(5.7), we conclude that H_{i+1} is well-defined and $\|H_{i+1}\|$ are uniformly bounded. Note that H_{i+1} is homogeneous of degree m. Then Q_{i+1} and H_{i+1} satisfy the required conditions. q.e.d.

Lemma 5.4. *Given any $m \ge 2$, we can find constants $C(m) > 0$ and $r_1 \ge r_m > 0$ and families of holomorphic maps $T_{i,m}$ from $D(r_m) \subset \mathbb{C}^n_i$ to $D(r_m) \subset \mathbb{C}^n_i$ and $G_{i+1,m}$ from $\mathbb{C}^n_i$ to $\mathbb{C}^n_{i+1}$ with the following properties:*

(i) *For each i, $T_{i+1,m}$ is a polynomial map of degree $m-1$ which is biholomorphic to its image, $T_{i+1,m}(0) = 0$, $T'_{i+1,m}(0) = Id$ and $\|T_{i+1,m}\| \le C(m)$.*

(ii) *$G_{i+1,m} = A_{i+1} + \widetilde{G}_{i+1,m}$ where $\widetilde{G}_{i+1,m}$ is a polynomial map of degree $m-1$,*

$$\begin{aligned} &\widetilde{G}_{i+1,m}(v_1 \oplus \cdots \oplus v_l) \\ &\quad = 0 \oplus \widetilde{G}_{i+1,m,2}(v_1) \oplus \cdots \oplus \widetilde{G}_{i+1,m,2}(v_1 \oplus \cdots \oplus v_{l-1}) \end{aligned}$$

is lower triangular, and $\|G_{i+1,m}\| \le C(m)$, $\widetilde{G}_{i+1,m}(0) = 0$ and $\widetilde{G}'_{i+1m}(0) = 0$. Moreover, $G_{i+1,m} = G_{i+1,m_0}$ for all $m \ge m_0$, where m_0 is the integer in Lemma 5.3.

(iii) $F_{i+1}(D(r_m)) \subset D(r_m)$ *and*

$$|T_{i+1,m}F_{i+1}(v) - G_{i+1,m}T_{i,m}(v)| \le C(m)|v|^m$$

in $D(r_m)$. *Here* $T_{i+1,m}F_{i+1} - G_{i+1,m}T_{i,m}$ *means* $T_{i+1,m} \circ F_{i+1} - G_{i+1,m} \circ T_{i,m}$.

Proof. Note that since A_{i+1} is nonsingular, $G_{i+1,m}$ will be a biholomorphism. We will construct the maps by induction. For $m = 2$, let $T_{i+1,m} = Id$, $G_{i+1,m} = A_{i+1}$. Since $F_{i+1}(D(r_1)) \subset D(r_1)$ and is holomorphic, by (5.2) we can take $r_2 = \frac{1}{2}r_1$, then it is easy to see that one can find $C(2)$ satisfies the required conditions. Suppose we have found $T_{i+1,m}$, $G_{i+1,m}$, $C(m)$ and r_m which have the required properties. Since

$$|T_{i+1,m}F_{i+1}(v) - G_{i+1,m}T_{i,m}(v)| \le C(m)|v|^m$$

we have $\|\Phi_{i+1}\| \le C_1$ for some C_1 which is independent of i, where Φ_{i+1} is the homogeneous polynomial of degree m which is the m-th power terms of the Taylor series of $T_{i+2,m}F_{i+1} - G_{i+1,m+1}T_{i+1,m}$. By Lemma 5.3, we can find H_{i+1} and Q_{i+1} such that both are homogeneous of degree m, H_{i+1} and Q_{i+1} satisfies conditions (i)–(iii) in Lemma 5.3 and

$$\Phi_{i+1} = Q_{i+1} + H_{i+1} - A_{i+2}^{-1}H_{i+2}A_{i+1}.$$

Now define $T_{i,m+1} = T_{i,m} + A_{i+1}^{-1}H_{i+1}$ and $G_{i+1,m+1} = G_{i+1,m} + Q_{i+1}$. Note that if $m \ge m_0$, then $Q_{i+1} = 0$. By the induction hypothesis, Lemma 5.3 and (5.3), it is easy to see that $T_{i+1,m+1}$ and $G_{i+1,m+1}$ satisfy (i) and (ii) of the lemma for some constants $C(m+1)$ and $r_{m+1} \le \frac{1}{2}r_m$. It remains to check condition (iii). We proceed as in [**36**].

In the following, $O(m+1)$ will denote some function h such that $|h(v)| \le C|v|^{m+1}$ for $|v| \le \frac{1}{2}r_m$, where C is a constant independent of i.

(5.8)

$$\begin{aligned}
&T_{i+1,m+1}F_{i+1} - G_{i+1,m+1}T_{i,m+1}\\
&= (T_{i+1,m} + A_{i+2}^{-1}H_{i+2})F_{i+1} - (G_{i+1,m} + Q_{i+1})(T_{i,m} + A_{i+1}^{-1}H_{i+1})\\
&= [T_{i+1,m}F_{i+1} - G_{i+1,m}T_{i,m}] + G_{i+1,m}T_{i,m}\\
&\quad - G_{i+1,m}(T_{i,m} + A_{i+1}^{-1}H_{i+1})\\
&\quad - Q_{i+1}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + A_{i+2}^{-1}H_{i+2}F_{i+1}.
\end{aligned}$$

Since $F_i\left(D\left(r_m\right)\right) \subset D\left(r_m\right)$, and $\|T_{i,m}\|$ and $\|G_{i,m}\|$ are uniformly bounded,

$$\begin{aligned}
&T_{i+1,m}F_{i+1} - G_{i+1,m}T_{i,m}\\
&= \Phi_{i+1} + O(m+1)\\
&= Q_{i+1} + H_{i+1} - A_{i+2}^{-1}H_{i+2}A_{i+1} + O(m+1).
\end{aligned}$$

Combining this with (5.8), we have

$$
\begin{aligned}
(5.9)\qquad & T_{i+1,m+1}F_{i+1} - G_{i+1,m+1}T_{i,m+1} \\
&= Q_{i+1} + H_{i+1} - A_{i+2}^{-1}H_{i+2}A_{i+1} + G_{i+1,m}T_{i,m} \\
&\quad - G_{i+1,m}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) \\
&\quad - Q_{i+1}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + A_{i+2}^{-1}H_{i+2}F_{i+1} + O(m+1) \\
&= \left[G_{i+1,m}T_{i,m} - G_{i+1,m}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + H_{i+1}\right] \\
&\quad + \left[Q_{i+1} - Q_{i+1}(T_{i,m} + A_{i+1}^{-1}H_{i+1})\right] \\
&\quad + \left[A_{i+2}^{-1}H_{i+2}F_{i+1} - A_{i+2}^{-1}H_{i+2}A_{i+1}\right] \\
&\quad + O(m+1).
\end{aligned}
$$

Denote the differential of a map h by h'. Then

$$
\begin{aligned}
& H_{i+2}F_{i+1} - H_{i+2}A_{i+1} \\
&= \int_0^1 \frac{d}{ds}(H_{i+1}(sF_{i+1} - (1-s)A_{i+1})ds \\
&= \int_0^1 \left[H'_{i+1}(sF_{i+1} - (1-s)A_{i+1})\right](F_{i+1} - A_{i+1})ds
\end{aligned}
$$

where the multiplication of the terms under the last integral sign is matrix multiplication. By (5.2), (5.3), the definition of A_{i+1} and the fact that $\|H_{i+1}\|$ are uniformly bounded and homogeneous of degree $m \geq 2$, we have

$$
(5.10)\qquad H_{i+2}F_{i+1} - H_{i+2}A_{i+1} = O(m+1).
$$

Using (5.3) and the facts that $\|Q_{i+1}\|$, $\|T_{i,m}\|$ and $\|H_{i+1}\|$ are uniformly bounded, Q_{i+1} is homogeneous of degree $m \geq 2$ and that $T'_{i,m}(0) = Id$, we can prove similarly that

$$
(5.11)\qquad Q_{i+1} - Q_{i+1}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) = O(m+1).
$$

Finally,

$$
\begin{aligned}
& G_{i+1,m}T_{i,m} - G_{i+1,m}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + H_{i+1} \\
&= -\int_0^1 \frac{d}{ds}\left(G_{i+1,m}(T_{i,m} + sA_{i+1}^{-1}H_{i+1})\right)ds + H_{i+1} \\
&= -\int_0^1 \left(\left[G'_{i+1,m}(T_{i,m} + sA_{i+1}^{-1}H_{i+1})\right](A_{i+1}^{-1}H_{i+1}) - A_{i+1}A_{i+1}^{-1}H_{i+1}\right)ds \\
&= -\int_0^1 \left(\left[G'_{i+1,m}(T_{i,m} + sA_{i+1}^{-1}H_{i+1}) - A_{i+1}\right](A_{i+1}^{-1}H_{i+1})\right)ds.
\end{aligned}
$$

Again the multiplication of the terms under the last two integral signs are matrix multiplications. Using (5.3) and the facts that $G'_{i+1,m}(0) = A_{i+1}$, that $\|G_{i+1,m}\|$, $\|H_{i+1}\|$ are uniformly bounded, and that H_{i+1} is homogeneous of degree m we conclude that

$$G_{i+1,m}T_{i,m} - G_{i+1,m}(T_{i,m} + A_{i+1}^{-1}H_{i+1}) + H_{i+1} = O(m+1). \tag{5.12}$$

From (5.9)–(5.12), we conclude that

$$|T_{i+1,m+1}F_{i+1}(v) - G_{i+1,m+1}T_{i,m+1}(v)| \le C(m+1)|v|^{m+1}.$$

This completes the proof of the lemma. q.e.d.

Let $m \ge m_0$ and denote $G_{i+1,m}$ simply by G_{i+1} and denote $\widetilde{G}_{i+1,m}$ by $\widetilde{G}_{i+1}$ etc. Note that G_{i+1} is independent of m and is a biholomorphism on $\mathbb{C}^n$ and that the degree of each G_{i+1} is $m-1$. For any positive integers i, j, let $G(i+j, i) = G_{i+j} \cdots G_{i+1}$.

Lemma 5.5. *Let G_{i+1} as above, then its inverse is a polynomial map of degree $(m-1)^{l-1}$ and satisfies:*

$$G_{i+1}^{-1} = A_{i+1}^{-1} + S_{i+1}$$

where $S_{i+1} : \mathbb{C}^n_{i+1} \to \mathbb{C}^n_i$ with

$$\begin{aligned} &S_{i+1}(w_1 \oplus \cdots \oplus w_l) \\ &= 0 \oplus S_{i+1,2}(w_1) \oplus \cdots \oplus S_{i+1,l}(w_1 \oplus \cdots \oplus w_{l-1}). \end{aligned}$$

Moreover, $\|G_{i+1}^{-1}\|$ is bounded by a constant independent of i.

Proof. Let $w_1 \oplus \cdots \oplus w_l \in E_{i+1,1} \oplus \cdots \oplus E_{i+1,l} = \mathbb{C}^n_{i+1}$. Let $v_1 = A_{i+1}^{-1}w_1$, $v_2 = A_{i+1}^{-1}(w_2 - \widetilde{G}_{i+1,2}(v_1)), \dots, v_l = A_{i+1}^{-1}(w_l - \widetilde{G}_{i+1,l}(v_1 \oplus \cdots \oplus v_{l-1}))$. Let $S_{i+1,k}(w_1 \oplus \cdots \oplus w_{k-1}) = -A_{i+1}^{-1}\widetilde{G}_{i+1,k}(v_1 \oplus \cdots \oplus v_{k-1})$, $2 \le k \le l$. It is easy to see that $S_{i+1,k}$ is well-defined and $S_{i+1,k}(w_1 \oplus \cdots \oplus w_{k-1}) \in E_{i,k}$ because $A_{i+1}(E_{i,k}) = E_{i+1,k}$. Moreover, the degree of each $S_{i+1,k}$ is at most $(m-1)^{k-1}$. It is also easy to see that

$$G_{i+1}^{-1} = A_{i+1}^{-1} + S_{i+1}$$

where $S_{i+1} = 0 \oplus S_{i+1,2} \oplus \cdots \oplus S_{i+1,l}$.

Let $w_1 \oplus \cdots \oplus w_l \in \mathbb{C}^n_{i+1}$ with $|w_k| \le 1$ and $v_1 \oplus \cdots \oplus v_l = G_{i+1}^{-1}(w_1 \oplus \cdots \oplus w_l)$. We claim that $|v_k|$ is bounded by a constant independent of i for each k. If this is true, then by Lemma 5.1 and (5.3) again, we can conclude that $\|G_{i+1}^{-1}\|$ is bounded by a constant independent of i. To prove the claim, by (5.3) have $|v_1| = |A_{i+1}^{-1}(w_1)|$ is uniformly bounded for $|w_1| \le 1$. Since $\|G_{i+1}\|$ is uniformly bounded by a constant independent of i, the same is true for $\|\widetilde{G}_{i+1,k}\|$ by Lemma 5.1(ii) and (5.3). Now suppose we have proved that $|v_1|, \dots, |v_{k-1}|$ are bounded by a constant independent of i. Then it follows that

$$|S_{i+1,k}(w_1 \oplus \cdots \oplus w_{k-1})| = |A_{i+1}^{-1}\widetilde{G}_{i+1,k}(v_1 \oplus \cdots \oplus w_{k-1})|$$

and hence $|v_k|$ are also bounded by a constant independent of i. The proof of the lemma then follows by induction. q.e.d.

Lemma 5.6. *Let $D(1)$ be the unit ball in $\mathbb{C}^n$ with center at the origin. Then the following are true:*

(i) *There exist $\beta > 0$ such that for all $z, z' \in D(1)$ and for any positive integers i and j,*

$$|G(i+j,i)^{-1}(z) - G(i+j,i)^{-1}(z')| \leq \beta^j |z - z'|.$$

(ii) *For any positive integer i and for any open set U containing the origin,*

$$\bigcup_{j=1}^{\infty} G(i+j,i)^{-1}(U) = \mathbb{C}^n.$$

Proof.

(i) Let us first assume that $i = 0$. Let us write

$$G(j,0)^{-1} = G_1^{-1} \cdots G_j^{-1} = H_{j,1} \oplus \cdots \oplus H_{j,l}$$

with $H_{j,k}(v) \in E_{1,k}$. By Lemma 5.2 and the Schwartz lemma, it is sufficient to prove that

$$|H_{j,k}(v)| \leq \beta^j \tag{5.13}$$

for some constant β and for all k and j provided $|v| \leq 1$. By Lemma 5.5, $G_i^{-1} = A_i^{-1} + S_i$ where S_i satisfies the conclusions in the lemma. Let $v = v_1 \oplus \cdots \oplus v_l \in \mathbb{C}_j^n$. Then $|H_{j,1}(v)| = |A_j^{-1} \cdots A_1^{-1}(v_1)| \leq a^j |v_1| \leq 2a^j$, where we have used Lemma 5.1(ii). Hence (5.13) is true for $k = 1$. Suppose (5.13) is true for $1, \ldots, k-1$. We may assume that $\beta > a^{-1}$. By Lemma 5.2 and 5.5, we know that $\|S_j\|$ is uniformly bounded. Let $C_j = \max_k \{\max_{|v| \leq 1} |H_{j,k}(v)|, 1\}$. Since $G_{j,k}^{-1}(w) = A_j^{-1}(w_k) + S_j(w_1 \oplus \cdots \oplus w_{k-1})$, we have

$$\begin{aligned} C_j &\leq a^{-1} C_{j-1} + C\beta^{(j-1)N} \\ &\leq 2C_{j-1}\beta_1^{j-1} \end{aligned}$$

where $N = (m-1)^{l-1}$ which is the degree of S_i, $C > 1$ is a constant dependent only on $\|S_i\|$ and N, and $\beta_1 = C\beta^N \geq a^{-1}$, where we have used the fact that $C_{j-1} \geq 1$. Hence $C_j \leq (2\beta_1)^{j-1} C_1$. From this the lemma follows for $i = 0$. For general i, the proof is similar. Note that the constants in the proof do not depend on i.

(ii) The proof is similar to the proof of (i). We only prove the case that $i = 0$ and the other cases are similar. Let us write $G_j \cdots G_1 = K_{j,1} \oplus \cdots \oplus K_{j,l}$. Then $K_{j,1}(v_1 \oplus \cdots \oplus v_l) = A_j \cdots A_1(v_1)$. Hence $K_{j,1}(v)$ converge to zero uniformly on compact sets. Suppose $K_{j,1}, \ldots, K_{j,k-1}$

converge uniformly to 0 on compact sets. Let Ω be a compact set and let $s_j = \sup_{v\in\Omega} |K_{j,k}|$. Then as before,

$$s_j \le b s_{j-1} + \sup_{v\in\Omega} |\widetilde{G}_{j,k}(K_{j-1,1}(v), \dots, K_{j-1,k-1}(v))|.$$

Hence

$$\limsup_{j\to\infty} s_j \le b \limsup_{j\to\infty} s_{j-1}$$

because $\|\widetilde{G}_{j,k}\|$ are uniformly bounded with uniformly bounded degrees and $K_{j-1,p}(v) \to 0$ uniformly on Ω for $1 \le p \le k-1$. From this it is easy to see that $s_j \to 0$ as $j \to \infty$. Hence $G_j \cdots G_1 \to 0$ uniformly on compact sets. From this (ii) follows. q.e.d.

Let β be the constant in Lemma 5.6. Note that β does not depend on i and m provided $m \ge m_0$, where $m_0 \ge 2$ is the integer in Lemma 5.3. This is because $G_{i,m} = G_{i,m_0}$ for all $m \ge m_0$. Fix $m \ge m_0$ such that

$$\delta^m \le \frac{1}{2}\beta^{-1} \tag{5.14}$$

where $1 > \delta > 0$ be the constant in (5.2). Let $G_{i,m}$, $T_{i,m}$ be the maps given in Lemma 5.4 which are defined on $D(r_m)$, $0 < r_m < r_1 < 1$. Let us denote $G_{i,m}$ by G_i, $T_{i,m}$ by T_i and r_m by r.

In the following, a holomorphic map Φ from a complex manifold to another is said to be *nondegenerate* if it is injective and so that it is a biholomorphism onto its image. We apply the method in [**36**] to obtain the following.

Lemma 5.7. *Let $k \ge 0$ be an integer. Then*

$$\Psi_k = \lim_{l\to\infty} G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}$$

exists and is a nondegenerate holomorphic map from $D(r)$ into $\mathbb{C}^n$. Moreover, there is a constant $\gamma > 0$ which is independent of k such that

$$\gamma^{-1} D(r) \subset \Psi_k(D(r)) \subset \gamma D(r). \tag{5.15}$$

Proof. Let $\Theta_l = G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}$. By the construction in Lemma 5.4, Θ_l is a nondegenerate holomorphic map on $D(r)$ and $\Theta_l(0) = 0$. For any $z \in D(r)$, let $w = F_{k+l} \circ \cdots \circ F_{k+1}(z)$. Then $|w| \le \delta^l r$ by (5.2). Hence $T_{k+l}(w)$, $T_{k+l+1} \circ F_{k+l+1}(w)$, $G_{k+l+1}^{-1} \circ T_{k+l+1} \circ F_{k+l+1}(w)$, and $G_{k+l+1} \circ T_{k+l}(w)$ are all in $D(1)$ for $l \ge l_0$ for some l_0 depending only on δ and m by Lemmas 5.4 and 5.5.

By Lemmas 5.4(iii) and 5.6, we have

$$\begin{aligned}
&\Big|G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ G_{k+l+1}^{-1} \circ T_{k+l+1} \circ F_{k+l+1} \circ F_{k+l} \cdots \circ F_{k+1}(z) \\
&\qquad - G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+1}(z)\Big| \\
&= \Big|G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ G_{k+l+1}^{-1} \circ T_{k+l+1} \circ F_{k+l+1}(w) \\
&\qquad - G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l}(w)\Big| \\
&\le \beta^l \Big|G_{k+l+1}^{-1} \circ T_{k+l+1} \circ F_{k+l+1}(w) - T_{k+l}(w)\Big| \\
&\le \beta^{l+1} \Big|T_{k+l+1} \circ F_{k+l+1}(w) - G_{k+l+1} \circ T_{k+l}(w)\Big| \\
&\le C_1 \beta^{l+1} |w|^m \\
&\le C_1 \beta^{l+1} \delta^{lm} \\
&\le C_1 \beta \left(\frac{1}{2}\right)^l
\end{aligned}$$

by (5.14) for some constant C_1 independent of k and l. From this it is easy to see that $\Psi_k = \lim_{l\to\infty} \Phi_l$ exists and is holomorphic on $D(r)$. Moreover

$$|\Psi_k(z)| \le |\Theta_{l_0}(z)| + C_1\beta.$$

Using (5.2) and the fact that $\|G_i\|$ and $\|T_i\|$ are uniformly bounded, we can find $\gamma > 1$ independent of k and l such that $\Psi_k(D(r)) \subset \gamma D(r)$. Since $\Phi_l'(0) = Id$, $\Psi_k'(0) = Id$. By the gradient estimates of holomorphic functions, $|\Phi_k'(z) - Id| \le C_2|z|$ on $\frac{1}{2}D(r)$ for some constant C_2 independent of k. Hence there exists $r > r' > 0$ independent of k such that Φ_k is nondegenerate in $D(r')$ and $\Psi_k(D(r)) \supset \gamma^{-1}D(r)$ provided γ is large enough independent of k. To prove that Ψ_k is nondegenerate on $D(r)$, let l_1 be such that $F_{k+l_1} \cdots \circ F_{k+1}(D(r)) \subset D(r')$. Then

$$\Psi_k = G_{k+1}^{-1} \circ \cdots \circ G_{k+l_1}^{-1} \circ \Psi_{k+l_1} \circ F_{k+l_1} \circ \cdots F_{k+1}.$$

Since $F_{k+l_1} \cdots \circ F_{k+1}$ is nondegenerate on $D(r)$, Ψ_{k+l_1} is nondegenerate on $D(r')$, and $G_{k+1}^{-1} \circ \cdots \circ G_{k+l_1}^{-1}$ is a biholomorphism of $\mathbb{C}^n$, we conclude that Ψ_k is nondegenerate on $D(r)$. q.e.d.

Now we are ready to prove the following uniformization theorem.

Theorem 5.1. *Let $(M^n, \widetilde{g})$ be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Suppose the scalar curvature of M satisfies*

$$\frac{1}{V_x(r)} \int_{B_x(r)} R \leq \frac{C}{1+r^2} \tag{5.16}$$

for some constant C for all $x \in M$ for all r. Suppose (M, g) has maximal volume growth. Then M is biholomorphic to $\mathbb{C}^n$. Moreover, the assumption of maximal volume growth can be removed if M has positive curvature operator.

Proof. If $\widetilde{g}$ satisfies the given conditions, then one can solve the Kähler-Ricci flow (2.3) and construct Φ_t and F_i as in the beginning of this section. We can also construct G_i, T_i as in Lemma 5.3 so that Lemmas 5.6 and 5.7 are true. Let $\Omega_i = \Phi_{iT}(D(r))$ where $r > 0$ is the constant in Lemma 5.7. By (5.1) and the fact that the solution $g(t)$ of (2.3) decays exponentially, $\{\Omega_i\}_{i \geq 1}$ exhausts M. Consider the following holomorphic maps from Ω_i to $\mathbb{C}^n$:

$$S_i = G_1^{-1} \circ \cdots G_i^{-1} \circ T_i \circ \Phi_{iT}^{-1}.$$

For each fixed k, and $l \geq 1$

$$\begin{aligned} &S_{k+l} \\ &= G_1^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ \Phi_{(k+l)T}^{-1} \\ &= G_1^{-1} \circ \cdots \circ G_k^{-1} \circ \left[G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+1}\right] \circ \Phi_{kT}^{-1}. \end{aligned}$$

By Lemma 5.7, we conclude that $S = \lim_{i \to \infty} S_i$ exists and is a nondegenerate holomorphic map from M into $\mathbb{C}^n$. Moreover, $S = G_1^{-1} \circ \cdots \circ G_k^{-1} \circ \Psi_k \circ \Phi_{kT}^{-1}$ on Ω_k where Ψ_k is the nondegenerate holomorphic map in Lemma 5.7. Hence

$$S(\Omega_k) = G_1^{-1} \circ \cdots \circ G_k^{-1} \circ \Psi_k(D(r)) \supset G_1^{-1} \circ \cdots \circ G_k^{-1}(\gamma^{-1} D(r))$$

by Lemma 5.7, for some γ independent of k. Therefore $S(M) = \mathbb{C}^n$ by Lemma 5.6(ii). This completes proof of the theorem. q.e.d.

By a recent result of Ni [**30**], if M has maximal volume growth, then (5.16) is satisfied automatically. Hence we have:

Corollary 5.1. *Let $(M^n, \widetilde{g})$ be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Suppose M has maximal volume growth, then M is biholomorphic to $\mathbb{C}^n$.*

We also have the following uniformization theorem.

Theorem 5.2. *Let $(M^n, \widetilde{g})$ be a complete noncompact Kähler manifold with nonnegative curvature operator such that the scalar curvature R of M satisfies* (5.16)*. Then the universal cover of M is biholomorphic to $\mathbb{C}^n$.*

Proof. Let $\tilde{g}(t)$ be the corresponding solution to the Kähler-Ricci flow 2.1. Let $\widetilde{M}$ be the universal cover of M. We then lift the flow $\tilde{g}(t)$ to $\widetilde{M}$ and denote the lifted flow by $\tilde{h}(t)$.

By the result in [**5**] and the De Rham decomposition theroem, one may assume that $\widetilde{M} = \mathbb{C}^k \times N_1 \times \cdots \times N_l$ isometrically and holomorphically so that each N_j is irreducible and has nonnegative curvature operator and positive Ricci curvature. Note that the flow $\tilde{h}(t)$ still satisfies the Kähler-Ricci flow equation when restricted on each N_j. Now suppose there is a positive constant C such that for t large enough, the injectivity radius of $\tilde{h}(t)$ is bounded below by $Ct^{1/2}$. Then by the proof of Theorem 5.1, it is not hard to show that in this case we can still have the results of sections §3, §4 and §5 for the restriction of $\tilde{h}(t)$ to any N_j, thus proving Theorem 5.2. We now proceed to show the above injectivity radius bound.

We claim that each N_j is noncompact. In fact, by the curvature assumption on M, there exists u such that $\sqrt{-1}\partial\overline{\partial}u = Ric_M$; see [**32**]. Let $\widetilde{u}$ be the lift of u to $\widetilde{M}$. Then $\sqrt{-1}\partial\overline{\partial}\widetilde{u} = Ric_{\widetilde{M}}$. In particular, $\widetilde{u}$ is strictly plurisubharmonic on each N_j. Hence N_j is noncompact.

By the proof in [**13**, pp. 25–26], one can conclude that for any $t_0 > 0$, there is a $\delta > 0$ such that $h(t)$ has positive sectional curvature for $t_0 < t \leq t_0 + \delta$ when restricted to N_j. Using the result of Gromoll-Meyer as before and using the fact that the curvature of N_j is bounded above by $C_1 t^{-1}$ by Theorem 2.1, one can conclude that the injectivity radius of $\tilde{h}(t)$ on N_j is bounded below by $C_1 t^{1/2}$ for some constant $C_1 > 0$ independent of t, t_0 and j. From this we can conclude that the injectivity radius of $h(t_0)$ on N_j is bounded below by $C_1 t^{1/2}$. Hence the injectivity radius of $h(t)$ on $\widetilde{M}$ is bounded below by $Ct^{\frac{1}{2}}$ for some constant $C > 0$ independent of t. This completes the proof of the theorem. q.e.d.

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